Real Analysis Preliminary Examination

August, 2025

DIRECTIONS: Complete seven (7) of the following nine problems, and indicate in the box below which seven problems should be graded.

If you do not do this, then problems 1-7 will be graded. Strive for clear, detailed, and legible solutions.

PROBLEMS

1. Let μ^* be an outer measure on X and $\{E_n\}_1^\infty$ be a sequence of subsets of X (not necessarily μ^*-measurable) such that $\sum_{n=1}^\infty \mu^*(E_n) < \infty$. Prove that both $E = \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty E_n$ and $F = \bigcup_{k=1}^\infty \bigcap_{n=k}^\infty E_n$ are μ^*-measurable.

2. Let $(X.\mathcal{M}, \mu)$ be a finite measure space and $\{f_n\}_1^\infty$ be a sequence of measurable functions such that $\lim_{n\to\infty} f_n = 0$ a.e. Let

$$E_n = \{x : |f_k(x)| \le 1 \quad \forall k \ge n\} \qquad \text{and} \qquad E = \{x : \lim_{n\to\infty} f_n(x) = 0\}$$

   Prove that

$$E \subset \bigcup_{n=1}^\infty E_n \qquad \text{and} \qquad \lim_{n\to\infty} \mu(E_n^c) = 0.$$

3. Let $(X.\mathcal{M}, \mu)$ be a measure space and $f : X \to [-\infty, \infty]$ be $\mathcal{M}$-measurable. Prove that if f is integrable, then $|f| < \infty$ a.e.

4. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $f \in L^1(X, \mathcal{M}, \mu)$. Prove that

$$\lim_{n\to\infty} \int_X |f|^{1/n} \, d\mu = \mu(\{x \in X : f(x) \ne 0\}).$$

5. Let $f, g \in L^1(\mathbb{R}, \mathcal{L}, m)$, and

$$f * g(x) = \int f(y-x) g(y) \, dm(y).$$

   Prove that $f * g \in L^1(\mathbb{R}, \mathcal{L}, m)$ and

$$\int f * g \, dm = \int f \, dm \int g \, dm.$$

6. Let $(X, \mathcal{M}, \nu)$ be a signed measure space. Prove that for any $E \in \mathcal{M}$

$$|\nu|(E) = \sup \{|\nu(A)| + |\nu(B)| : \; A, B \in \mathcal{M}, \quad E = A \cup B, \quad A \cap B = \emptyset\}.$$

7. Let X be a normed space and $x \in X$ be a nonzero vector. Prove that there is a bounded linear functional f such that
$$X = \ker(f) \oplus \mathrm{span}(x).$$

8. Let E be a closed subspace of a Hilbert space H and $\{v_i : i \in \mathbb{N}\}_{\alpha \in A}$ be an orthonormal basis for E. Prove that for any $x \in H$, $z = \sum_{i=1}^{\infty} \langle x, v_i \rangle v_i$ converges in E, and z is the unique vector in E closest to x.

9. Let $g \in L^{\infty}(\mathbb{R}, \mathcal{L}, m)$ and $1 \leq p < \infty$. Define a linear operator T on $L^p(\mathbb{R}, \mathcal{L}, m)$ by $T(f) = fg$. Show that T is a bounded linear operator on $L^p(\mathbb{R}, \mathcal{L}, m)$ and $\|T\| = \|g\|_{\infty}$.