Real Analysis Preliminary Examination

May, 2025

DIRECTIONS: Complete seven (7) of the following nine problems, and indicate in the box below which seven problems should be graded.

If you do not do this, then problems 1-7 will be graded. Strive for clear, detailed, and legible solutions.

PROBLEMS

1. Let μ^* be an outer measure over X and A be a μ^*-measurable set. Prove that if a set $E \subset X$ has the property that $\mu^*(E \ \ A) = 0$, then E is μ^*-measurable.

2. Let m^* be the Lebesgue outer measure on $\mathcal{P}(\mathbb{R})$ and m be the Lebesgue measure. Prove that $\forall E \subset \mathbb{R}$, $\exists$ a Borel set $B \subset \mathbb{R}$ such that $B \supset E$ and $m(B) = m^*(E)$.

3. Let $(X, \mathcal{M}, \mu)$ be a complete measure space and $f, g : X \to [-\infty, \infty]$ such that $f = g$ a.e. If f is $\mathcal{M}$-measurable, prove that g is also $\mathcal{M}$-measurable.

4. Compute
$$\lim_{n\to\infty} \int_0^\infty \frac{x}{\sqrt{1+x^n}}\, dx$$
and justify all your steps.

5. Let f be a Lebeqgue integrable function over $\mathbb{R}$. Prove that $\int_0^\infty \int_{\ln y}^\infty e^{-x} f(x)\, dxdy$ exists, and
$$\int_0^\infty \int_{\ln y}^\infty e^{-x} f(x)\, dxdy = \int_{-\infty}^\infty f(x)\, dx.$$

6. Let ν_1, ν_2, μ be finite signed measures such that $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$. Accepting the fact without proving that $a\nu_1 + b\nu_2$ is a signed measure for any $a, b \in \mathbb{R}$, prove that
$$(a\nu_1 + b\nu_2) \perp \mu.$$

7. Suppose $f \in C[a,b]$, $-\infty < a < b < \infty$, and for every nonnegative integer k, $\int_{[a,b]} f(x)x^{3k}\, dx = 0$. Show that $f \equiv 0$ on $[a,b]$.

8. Let X, Y be Banach spaces over $\mathbb{R}$, $L(X,Y)$ be the space of bounded linear operators from X to Y, and $\{T_n\} \subset L(X,Y)$ which converges pointwise on X to a function T. Show that $T \in L(X,Y)$.

9. Suppose $f \in L^p(X, \mathcal{M}, \mu)$ for all $1 \le p < \infty$ and that there exists a constant $c > 0$ such that $\|f\|_p \le c$ for all $1 \le p < \infty$. Prove that $f \in L^\infty(X, \mathcal{M}, \mu)$.