Real Analysis Preliminary Examination

May 2026

DIRECTIONS

Complete seven (7) of the following nine problems, and indicate in the box below which seven problems should be graded.

If you do not do this, then problems 1-7 will be graded. Strive for clear, detailed, and legible solutions.

On this exam, $(X, \mathcal{M}, \mu)$ denotes a measure space and m, m_n denote Lebesgue measure on $\mathbb{R}, \mathbb{R}^n$, respectively.

PROBLEMS

1. Let (A_n) be a sequence of measurable sets such that $\mu(A_n) < n^{-25/24}$ for all $n \in \mathbb{N}$. Show that
$$\mu\left(\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} A_j\right) = 0.$$

2. (a) (3 pts) State the Hahn-Banach Theorem in the case of a vector space with a sublinear functional.

   (b) (7 pts) Use Part (a) to prove that if X is a normed vector space and $f : M \longrightarrow \mathbb{R}$ is a bounded linear functional on a subspace M of X, then there exists a bounded linear functional $F : X \longrightarrow \mathbb{R}$ such that $F|_M = f$ and $\|F\| = \|f\|$.

3. If E is a Lebesgue measurable subset of $\mathbb{R}$ and $m(E) < \infty$, prove that for all $\epsilon > 0$ there exists a set A that is a finite union of disjoint open intervals such that $m(E \triangle A) < \epsilon$.

4. Prove that if $f, g : X \longrightarrow [0, \infty]$ are measurable, then fg is measurable. (Use the convention that $\infty \cdot 0 = 0 \cdot \infty = 0$.) *Hint:* Give a rational proof.

5. Compute
$$\lim_{n\to\infty} \int_0^{\infty} \frac{x}{1 + x^n}\, \mathrm{d}x,$$
and prove your answer.

6. Let A be an open subset of $\mathbb{R}^n$. For $h > 0$, define
$$B_h = \left\{(x_1, \dots, x_n, h) \in \mathbb{R}^{n+1} \ : \ (x_1, \dots, x_n) \in A\right\}, \qquad \text{and}$$
$$C = \left\{(\lambda x_1, \dots, \lambda x_n, \lambda z) \in \mathbb{R}^{n+1} \ : \ (x_1, \dots, x_n, z) \in B_h \quad \text{and } 0 \le \lambda \le 1\right\}.$$
Find $m_{n+1}(C)$ and prove your answer. You need not prove that C is measurable.

7. Let $(Y, \mathcal{N})$ be a measurable space and $\varphi : X \longrightarrow Y$ be a measurable mapping. Define $\nu(A) = \mu\left(\varphi^{-1}(A)\right)$ for all $A \in \mathcal{N}$. Show that ν is a measure and that $\int f \,\mathrm{d}\nu = \int (f \circ \varphi) \,\mathrm{d}\mu$ for all measurable $f : Y \longrightarrow [-\infty, \infty]$ such that the integral on the right-hand side is defined.

8. If ν is a signed measure, prove that E is ν-null if and only if $|\nu|(E) = 0$.

9. Let $f \in L^\infty[0,1]$. Prove that

$$\|f\|_\infty = \lim_{p \to \infty} \|f\|_p.$$