REAL ANALYSIS QUALIFYING EXAMINATION

TEXAS TECH UNIVERSITY 1996

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded.

1. Let f be an integrable function on a measure space $(X, \mathfrak{M}, \mu)$ with $\mu(X) < \infty$. Show that if $\int_E f d\mu \leq \mu(E)$ for all $E \in \mathfrak{M}$ then $f \leq 1$ almost everywhere.

2. Let $(X, \mathfrak{M}, \mu)$ be a $\sigma-$finite measure space. Let $T \subset \mathbb{R}$ be open and let $f : X \times T \to \mathbb{R}$ be a map satisfying:
   i) For each $t \in T$ the map $x \mapsto f(x,t)$ is in $L^1(X, \mu)$.
   ii) For each x, the map $f_x : t \mapsto f(x,t)$ is continuously differentiable on T
   iii)
   $$\frac{\partial f(x,t)}{\partial t} \in L^1(X, \mu)$$
   for all $t \in T$ and there exist $g \in L^1(X, \mu)$ with $g \geq 0$ and
   $$|\frac{\partial f(x,t)}{\partial t}| \leq g(x)$$
   for all $(x,t) \in X \times T$.

   Prove that the function
   $$\Phi(t) = \int_X f(x,t) d\mu(x)$$
   is differentiable and that
   $$\Phi'(t) = \int_X \frac{\partial f(x,t)}{\partial t} d\mu(x).$$

3. Let $H, \langle\ ,\ \rangle$ be a Hilbert Space and $A : H \to H$ a bounded linear operator. Show that there is a unique linear operator $A^* : H \to H$ such that $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x, y \in H$. Show that $\|A\| = \|A^*\|$.

4. Let μ_1 and ν_1 be σ-finite measures on $(X_1, \mathfrak{M}_1)$ and let μ_2 and ν_2 be σ-finite measures on $(X_2, \mathfrak{M}_2)$ such that $\nu_1 << \mu_1$ and $\nu_2 << \mu_2$. Show that $\nu_1 \times \nu_2 << \mu_1 \times \mu_2$ and that
   $$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2)$$

5. Let $K(x, y)$ be a continuous function on the unit square $[0,1] \times [0,1]$. Let $C^0([0,1])$ be the Banach space of continuous functions on $[0,1]$ with the "sup norm". Define a linear operator $T : C^0([0,1]) \to C^0([0,1])$ by
$$Tg(x) = \int_0^1 K(x,t)g(t)dt.$$
a) Show that T is continuous.
b) Show that T is compact; that is, for every bounded sequence $\{f_n\} \subset C^0([0,1])$ the sequence $\{Tf_n\}$ has a convergent subsequence. *Hint: Arzela- Ascoli.*

6. State and prove the Baire Category Theorem.

7. Let m be Lebesgue measure on $\mathbb{R}^n$. For $f \in L^1(\mathbb{R}^n, m)$ define a function A_f on $(0,\infty) \times \mathbb{R}^n$ by
$$A_f(r,x) = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)dm(y).$$
Show
a) that $A_f(r,x)$ is continuous in r for each fixed $x \in \mathbb{R}^n$ and
b) that $A_f(r,x)$ is measurable in x for each fixed $r > 0$.

8. Prove the following theorem (Egoroff's Theorem):

Theorem. Suppose $\mu(X) < \infty$, and that f, f_n, $n = 1, 2, \ldots$ are measurable functions and that $f_n \to f$ a.e. Then for every $\epsilon > 0$ there exists a measurable set $E \subset X$ such that $\mu(E) < \epsilon$ and such that $f_n \to f$ uniformly on $E^c = X - E$.

9. Let $(X, \mathfrak{M}, \mu)$ be a measure space with $\mu(X) < \infty$ and suppose a measurable function f is in $L^p(\mu)$ for all $p \geq 1$. Show that $f \in L^\infty$ and that
$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$
Hint: You may find it useful to prove the inequalities " $\limsup_{p\to\infty} \|f\|_p \leq \|f\|_\infty$" *and* " $\|f\|_\infty \leq \liminf_{p\to\infty} \|f\|_p$" *separately.*

10. Let $f \in L^p(\mathbb{R})$, where $p \geq 1$. Define $f_h(x) \equiv f(x+h)$. Show that $f_h \in L^p(\mathbb{R})$ and that $\lim_{h\to 0} \|f_h - f\|_p = 0$.