Real Analysis Preliminary Examination

1997

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded.

1. Let
$$f(x) = \begin{cases} e^{x^2} & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is an irrational number} \end{cases}.$$

   a. Show that f is Lebesgue measurable.

   b. Compute
$$\int_{\mathbf{R}} f(x)\, dx$$

2. Let $\mathcal{B}_{\mathbf{R}}$ be the Borel σ-algebra on $\mathbf{R}$. Show that every finite measure on $\mathcal{B}_{\mathbf{R}}$ is a Lebesgue-Stieltjes measure μ_F associated with some nondecreasing, right continuous function F.

3. Let f be a real-valued integrable function on a measure space $(X, \mathcal{M}, \mu)$, and $\{E_i\}_{i=1}^{\infty}$ be measurable subsets of X such that
$$\mu(\bigcap_{i=1}^{\infty} E_i) = 0.$$
Show that
$$\lim_{n\to\infty} \int_{\bigcap_{i=1}^{n} E_i} f\, d\mu = 0.$$

4. Let $f_n \geq 0$, $n = 1, 2, \ldots$, be a sequence of nonnegative measurable functions on a measure space $(X, \mathcal{M}, \mu)$ such that $f_n \to f$ in measure. Show that
$$\int_X f\, d\mu \leq \liminf \int_X f_n\, d\mu.$$

5. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two σ-finite measure spaces, and f be a $\mu\times\nu$-measurable function on $X \times Y$ such that
$$\int_Y \left[\int_X |f|\, d\mu\right] d\nu \leq \infty.$$
Show that

   a. f is integrable over the product measure $\mu \times \nu$.

   b.
$$\int_{X\times Y} f\, d(\mu\times\nu) = \int_Y \left[\int_X f\, d\mu\right] d\nu = \int_X \left[\int_Y f\, d\nu\right] d\mu.$$

6. Let ν be a signed measure on a measurable space $(X, \mathcal{M})$ and $X = P \cup N$ be a Hahn decomposition for ν.

a. Define the positive variation ν^+, negative variation ν^-, and total variation $|\nu|$, of ν.

b. Show that

$$|\nu(E)| \leq |\nu|(E) \text{ for each } E \in \mathcal{M},$$

c. Show that

$$|\nu(E)| = |\nu|(E)$$

if and only if either $\nu(E \cap P) = 0$ or $\nu(E \cap N) = 0$.

7. Let ν and μ be σ-finite **positive** measures on a measurable space $(X, \mathcal{M})$ with $\nu \ll \mu$. Show that the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ of ν with respect to μ satisfies

$$\frac{d\nu}{d\mu} \geq 0 \ \mu\text{-a.e.}$$

8. Let X be a normed vector space, $\{x_n\} \subset X$, and $\{f_n\} \subset X^*$ be a sequence of continuous linear functionals on X. Show that if $f_n \to f \in X^*$ and $x_n \to x \in X$ then

$$f_n(x_n) \to f(x).$$

9. Let $K(x,t)$ be a Lebesgue measurable function on $\mathbf{R}^2$ such that

$$M := \left(\iint_{\mathbf{R}^2} |K(x,t)|^2 \, dt dx\right)^{1/2} < \infty.$$

Show that the linear operator T defined by

$$T(f)(x) := \int_{\mathbf{R}} K(x,t) f(t) \, dt$$

is a bounded linear operator on $L^2(\mathbf{R})$ and

$$||T|| \leq M.$$

10. Let H be a Hilbert space and S be any subset of H. Show that

$$S^\perp := \{x \in H : \ < x, s >= 0 \text{ for all } s \in S\}$$

is a closed subspace.