Preliminary Examination **Real Analysis** **May 1999**

Do seven of the ten problems. You must indicate which seven problems are to be graded.

1. Let M be a finite dimensional subspace of a normed linear space X. Prove that there exists a closed subspace N such that $M \cap N = \{0\}$ and $M + N = X$.

2. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on a measure space X. Prove that $\{x | \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is measurable.

3. Let f be a Lebesgue integrable function on the real line $\Re$, and let $\epsilon > 0$. Prove that there is a continuous function g which vanishes outside a bounded interval such that $\int_{\Re} |f - g| dm < \epsilon$, where m denotes Lebesgue measure.

4. Compute the following and justify your answer.
(a) $\lim_{n\to\infty} \int_0^{\infty} \frac{n}{x(1+x^2)} \, sin(x/n) dx$.
(b) $\int_{-\infty}^{\infty} e^{-x^2} cos(ax) dx$ where $a > 0$.

5. Let $\{X, \mathcal{M}, \mu\}$ be a measure space and $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions such that $f_n \to f$ a.e. and $|f_n(x)| \leq |g(x)|$ a.e. x, $\forall n$ and for some $g \in L^1(\mu)$. Prove that $\forall\, \epsilon > 0$, $\exists A \in \mathcal{M}$ such that $\mu(A) < \epsilon$ and $f_n \to f$ uniformly on $X \backslash A$.

6. (a) Let $\{X, \mathcal{M}\}$ be a measure space, and μ and ν are measures on $\mathcal{M}$. Suppose that $\nu(X) < \infty$ and $\nu << \mu$. Show that $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ such that $\nu(E) < \epsilon$ whenever $\mu(E) < \delta$.
(b) Use (a) to prove the following: let $\{f_n\}$ be a sequence which converges in $L^1(\mu)$ to f. Then, $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ such that $\int_E |f_n| d\mu < \epsilon$ for all n whenever $\mu(E) < \delta$.

7. Find $f \in L^2[0,1]$ with *minimum* norm which satisfies the equation $\int_0^1 e^{1-t} f(t) dt = 1$. (Possible hint: Use Hilbert space theory.)

8. Let H be an infinite dimensional Hilbert space and $\{x_n\}_{n=1}^{\infty}$ be an orthonormal basis of H. Let f be a nonzero element in H.
(a) Show that $< f, x_n > \to 0$ as $n \to \infty$.
(b) Let $g \in H$ be such that $\|g\| < 1$. Prove that there exists a sequence $\{y_n\}_{n=1}^{\infty}$ such that $\|y_n\| = 1$ for all n and $< g, y_n - g > \to 0$ as $n \to \infty$.

9. Let $\mathcal{X}$ be a linear space and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on $\mathcal{X}$ such that $\|x\|_1 \leq \|x\|_2$ for all $x \in \mathcal{X}$. Suppose further that $(\mathcal{X}, \|.\|_1)$ and $(\mathcal{X}, \|.\|_2)$ are both complete. Prove that $\exists c > 0$ such that $\|x\|_2 \leq c\|x\|_1$ for all $x \in \mathcal{X}$.

10. Let $X = Y = [0,1]$ and m denote the Lebesgue measure on $[0,1]$. Let $K : X \times Y \to \Re$ be given by

$$K(x,y) = \begin{cases} \frac{1}{\sqrt{y-x}} & \text{if } x < y, \\ \frac{-2}{\sqrt{x-y}} & \text{if } y < x, \\ 0 & \text{otherwise.} \end{cases}$$

Evaluate $\int_{X\times Y} K \, d(m \times m)$. Be sure to verify the hypotheses of every theorem you use.