Real Analysis Examination 2000

You may answer as many problems as you can. The best seven out of ten will be used for scoring.

To receive credit you must justify your answers.

Following notation is used throughout:

(i) $\Re$ denotes the Real line.

(ii) m denotes the Lebesgue measure on $\Re$.

(iii) For subsets E and F of a set X denote by $E \Delta F$ the symmetric difference $(E \backslash F) \cup (F \backslash E)$.

(iv) l^2 is the Hilbert space of square summable real sequences $\{y_n\}_{n=1}^{\infty}$.

(v) e_n denotes that element of l^2 with a 1 in the n^{th} slot, all other elements equal to zero.

Problem 1: If $E \subset \Re$ has finite Lebesgue measure, prove that for all $\epsilon > 0$ there exists a set A which is a finite union of open intervals, such that $m(E \Delta A) < \epsilon$.

Problem 2: Use Fatou's Lemma to prove the monotone convergence theorem.

Problem 3: State and prove the closed graph theorem.

Problem 4: Let $(X, \mathcal{M})$ be a measure space and let μ and ν be finite measures on $\mathcal{M}$ such that $\nu \ll \mu$. Let $\lambda = \mu + \nu$.

(a) State the Radon-Nikodym theorem and use it to show that there exists a measurable function f, $0 < f(x) < 1$ a.e. with respect to μ, and such that $\nu(E) = \int_E f d\lambda$ for every measurable set E.

(b) Show that $\int_X g d\lambda = \int_X g d\mu + \int_X g d\nu$ for each measurable function g.

(c) Suppose there exists h, $0 \le h < 1$, such that $\int_X g d\nu = \int_X h g d\lambda$ for every nonegative measurable function g. Show that $\nu(E) = \int_E \frac{h}{1-h} d\mu$ for every measurable set E.

Problem 5: Let X be a Banach space, and $\{x_n\}_{n=1}^{\infty}$ a weakly convergent sequence in X. Show that $\{\|x_n\|\}_{n=1}^{\infty}$ is bounded.

Problem 6: Show that $\{e_n\}_{n=1}^{\infty}$ converges to zero weakly in l^2.

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Problem 7: Given that $f \in L^p \cap L^\infty$ for some $p < \infty$.

(a) Verify that $\|f\|_q \leq \|f\|_p^{(\frac{p}{q})} \|f\|_p^{(1-\frac{p}{q})}$ for all $p \leq q \leq \infty$.

(b) Use (a) to deduce that $f \in L^q$ for all $q \geq p$.

(c) Prove that $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$.

Problem 8: Consider the set $X = \{0, 1, 2\}$. Starting with the set-valued map $\phi : \{\emptyset, \{0\}, \{1\}, \{1, 2\}, \{0, 1, 2\}\} \to \Re$,

$$\begin{aligned}
\phi(\emptyset) &= 0, \\
\phi(\{0\}) &= 4, \\
\phi(\{1, 2\}) &= 2, \\
\phi(\{1\}) &= 3, \\
\phi(\{0, 1, 2\}) &= 3,
\end{aligned}$$

and using the Caratheodory extension, construct an outer measure μ^* on X. What are the μ^*-measurable subsets of X?

Problem 9: Evaluate $\int_0^\infty e^{-x^2} \cos(x) dx$.

Hints:

(1) Write $\cos(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{(2m)!}$.

(2) Use (without proof) the fact that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

Problem 10: Let $\alpha \in (0, 1)$ be a constant. For $k = 1, \cdots$, define x_k to be the sequence $\{1, \alpha^k, \alpha^{2k}, \alpha^{3k}, \cdots, \alpha^{(n-1)k}, \cdots\}$.

(a) Show that $x_k \in l^2$.

(b) Let M be the smallest closed subspace that contains all vectors x_k. Show that $M = l^2$.