Real Analysis Preliminary Examination

2001

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded. Notations: $\mathbb{C}$ =the set of complex numbers, $\mathbb{R}$ =the set of real numbers.

1. Let $\mathcal{S} = \{E_1, E_2, \ldots, E_n\}$ be a collection of nonempty subsets of X such that

$$\bigcup_{i=1}^{n} E_i = X \text{ and } E_i \cap E_j = \emptyset \text{ for } i \neq j.$$

   Find the σ-algebra generated by $\mathcal{S}$.

2. For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ be the largest integer less than or equal to x. Let

$$F(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ \lfloor x \rfloor, & \text{if } x > 0 \end{cases},$$

   and let μ_F be the Lebesgue-Stieltjes measure associated to F. Compute

$$\int 3^{-x} \, d\mu_F.$$

3. Let $\mathcal{B}_{\mathbb{R}}$ be the Borel σ-algebra on $\mathbf{R}$, and μ be a measure on $\mathcal{B}_{\mathbb{R}}$ which is finite on every bounded set in $\mathcal{B}_{\mathbb{R}}$. Define

$$F(x) = \begin{cases} \mu((0, x]), & \text{if } x \geq 0 \\ -\mu((x, 0]), & \text{if } x < 0 \end{cases}$$

   Show that

   a. F is increasing,

   b. F is right continuous,

   c. μ is the Lebesgue-Stieltjes measure associated to F.

4. Let f be a nonnegative measurable function on a measure space $(X, \mathcal{M}, \mu)$, and

$$E_1 \subset E_2 \subset \cdots$$

   be measurable subsets of X. Prove that

$$\int_{\bigcup_{n=1}^{\infty} E_n} f \, d\mu = \lim_{n \to \infty} \int_{E_n} f \, d\mu.$$

5. Let f_n, $n = 1, 2, \ldots$, be a sequence of integrable functions on a measure space $(X, \mathcal{M}, \mu)$ such that

$$\int |f_n| \, d\mu \leq M < \infty \text{ for all } n$$

and $f_n \to f$ in measure. Prove that f is integrable and

$$\int |f| \, d\mu \leq M.$$

6. Show that if a real series $\sum_{i,j=1}^{\infty} a_{ij}$ converges absolutely, then

$$\sum_{i,j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

7. Let ν be a signed measure on a measurable space $(X, \mathcal{M})$, and $X = P \cup N$ be a Hahn decomposition for ν. Prove that

   a. $|\nu(E)| \leq |\nu|(E)$ for all E in $\mathcal{M}$,

   b. $|\nu(E)| = |\nu|(E)$ if and only if either $\nu(E \cap P) = 0$ or $\nu(E \cap N) = 0$.

8. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $T : \mathcal{X} \to \mathcal{Y}$ be an injective bounded linear map, and $\mathcal{M}$ be the range of T. Prove that $T : \mathcal{X} \to \mathcal{M}$ is an isomorphism if and only if $\mathcal{M}$ is closed.

9. Show that in an inner product space over $\mathbb{C}$,

$$\langle x, y \rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2)$$

and use it to prove that there is at most one inner product which generates the same induced norm, namely $\|x\| = \sqrt{\langle x, x \rangle}$.

10. Let $k(x, t)$ be a Lebesgue measurable function on $\mathbb{R}^2$ such that

$$\left(\iint |k(x,t)|^q \, dt dx\right)^{1/q} < \infty, \text{ for } 1 < q < \infty$$

and let p satisfy

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Define

$$T(f)(x) = \int k(x,t) f(t) \, dt.$$

Prove that $T(f)$ is in $L^q(\mathbb{R})$ for every $f \in L^p(\mathbb{R})$, T is a bounded linear operator from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ and

$$\|T\| \leq \left(\iint |k(x,t)|^q \, dt dx\right)^{1/q}.$$