**Problem 1.** In each part, give the requested example. Clear pictures and descriptions of the functions are acceptable in place of explicit formulas. Be sure to give the justification that your example works.

A. Give an example of a sequence $\{ f_n \} \subseteq L^1(\mathbb{R})$ such that $f_n \to 0$ pointwise a.e., but

$$\int_{\mathbb{R}} f_n \, dm \not\to 0.$$

B. Give an example of a sequence $\{ f_n \} \subseteq L^1(\mathbb{R})$ such that $f_n \to 0$ **uniformly** but

$$\int_{\mathbb{R}} f_n \, dm \not\to 0?$$

C. Give an example of a function $f \in L^1(\mathbb{R})$ such that $f \notin L^2(\mathbb{R})$.

D. Give an example of a function $f \in L^2(\mathbb{R})$ such that $f \notin L^1(\mathbb{R})$.

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**Problem 2.** Suppose that $(X, \mathcal{M}, \mu)$ is a finite measure space (i.e., $\mu(X) < \infty$) and that $f \colon X \to [0, \infty]$ is a measurable function. Suppose that

$$\int_E f \, d\mu \le \mu(E)$$

for all measurable sets $E$. Show that $f \le 1$ $\mu$-a.e.

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**Problem 3**. Recall that if $f$ is Riemann integrable on $[a, b]$ then it is Lebesgue measurable on $[a, b]$ and

$$\int_a^b f(x)\,dx = \int_a^b f(x)\,dm(x),$$

where the left-hand side denotes the Riemann integral and the right-hand side denotes the Lebesgue integral.

Suppose that $f\colon [0, \infty) \to \mathbb{R}$ is Riemann integrable on every compact subinterval of $[0, \infty)$. In each part, state clearly which convergence theorems for Lebesgue integrals you are using.

A. Show that $f$ is Lebesgue measurable and

$$\int_0^\infty |f(x)|\,dx = \int_0^\infty |f(x)|\,dm(x),$$

(the left-hand side is an improper Riemann integral).

B. Assume that

$$\int_0^\infty |f(x)|\,dx < \infty.$$

Show that both of the integrals in the following equation exist and that the equation holds:

$$\int_0^\infty f(x)\,dx = \int_0^\infty f(x)\,dm(x).$$

C. Give an example where

$$\int_0^\infty f(x)\,dx$$

exists but

$$\int_0^\infty f(x)\,dm(x)$$

does **not** exist.

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**Problem 4**. Let $X$ be a topological space and let $f_n\colon X \to \mathbb{R}$ $(n = 1, 2, \dots)$ be a sequence of Borel measurable functions.

A. Show that the functions $g$ and $h$ defined by

$$g(x) = \sup \{\, f_n(x) \mid n = 1, 2, \dots \}$$
$$h(x) = \inf \{\, f_n(x) \mid n = 1, 2, \dots \}$$

are Borel measurable.

B. Suppose that the sequence $\{\, f_n \,\}$ converges pointwise to a function $f$. Use the previous part to show that $f$ is Borel measurable.

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**Problem 5**. Let $(X, \mathcal{M})$ be a measurable space and let $\mu$ and $\lambda$ be finite positive measures on $(X, \mathcal{M})$, so $\nu = \mu - \lambda$ is a signed measure. Show that $\nu^+ \leq \mu$ and $\nu^- \leq \lambda$.

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**Problem 6**.

A. State the Fubini-Tonelli Theorem for the product of two complete measure spaces.

B. Evaluate the integrals

$$\int_0^1 \left[ \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dx \right] dy \qquad \text{and} \qquad \int_0^1 \left[ \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dy \right] dx.$$

Hints: The answer may be $\pm\infty$. You may use without proof the formulas

$$\int_0^z \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1}(z/a)$$
$$\int_0^z \frac{du}{(a^2 + u^2)^2} = \frac{z}{2a^2(a^2 + z^2)} + \frac{1}{2a^3} \tan^{-1}(z/a).$$

C. Are the two integrals in part B equal? If so, does the equality follow from part A? If they are not equal, explain which of the hypotheses of the Fubini-Tonelli Theorem fail.

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**Problem 7**. Suppose that $f \in L^p(\mathbb{R})$, where $1 \le p < \infty$. For $t \in \mathbb{R}$, define $f_t$ by $f_t(x) = f(x-t)$. Show that $f_t \in L^p(\mathbb{R})$ for all $t$ and that

$$\lim_{t\to 0} \|f_t - f\|_p = 0.$$

Hint: Consider first the case where $f$ is a continuous function with compact support.

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**Problem 8**. Let $\mathcal{X}$ be a normed complex vector space and suppose $x_0 \in \mathcal{X}$, where $x_0 \neq 0$. Let $\mathcal{M} = \{\, \lambda x_0 \mid \lambda \in \mathbb{C} \,\}$ be the one dimensional subspace of $\mathcal{X}$ spanned by $x_0$. Show that there is a closed subspace $\mathcal{N} \subseteq \mathcal{X}$ such that $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$, i.e., every vector $x \in \mathcal{X}$ can be written uniquely in the form $x = \lambda x_0 + n$, where $\lambda \in \mathbb{C}$ and $n \in \mathcal{N}$. Hint: Hahn-Banach Theorem.

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**Problem 9**. Let $\mathcal{X}$ be a Banach space and let $\{\, f_n \,\} \subseteq \mathcal{X}^*$ be a sequence of continuous linear functionals. Suppose that for every $x \in X$, the sequence $\{\, f_n(x) \,\}$ converges to something, call it $f(x)$. Show that $f$ is a continuous linear functional on $X$.

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**Problem 10**. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$. Suppose that $1 \le r < s \le \infty$. Show that

$$L^s(\mu) \subseteq L^r(\mu)$$

and that the inclusion mapping $j \colon L^s(\mu) \hookrightarrow L^r(\mu) \colon f \mapsto f$ is bounded (equivalently, continuous). Hint: Hölder's Inequality.

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# EXAM

Preliminary Examination in Real Analysis

Summer 2002

- Write all of your answers on separate sheets of paper. You can keep the exam questions when you leave. You may leave when finished.
- There are 10 problems. **The best 7 scores will be added for your grade.**

Good luck!