Real Analysis Preliminary Examination

August 2003

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded. All theoretical results that you use must be mentioned.

1. Find the σ-algebra generated by the subsets $\{1,2\}$ and $\{2,3\}$ of the space $X = \{1,2,3,4,5,6\}$.

2. Let $(X, \mathcal{M})$ be a measurable space. Prove that if $f : X \to \mathbf{R}$ has the property that $f^{-1}((r,\infty))$ is measurable for any rational number r, then f is a measurable function.

3. Rigorously compute the integral

$$\int_0^1 \ln x \ln(1-x) dx$$

(Hint. Expand in Taylor series the function $\ln(1-x)$.)

4. Show that

$$\int_0^\infty x^n e^x dx = n!$$

by differentiating the equation

$$\int_0^\infty e^{-tx} dx = \frac{1}{t}.$$

Be sure that you verify the hypothesis of the theorems that you use.

5. Let $F, G : (0,10) \to \mathbf{R}$, $F(x) = x^3 + \lfloor 3x \rfloor$ and $G(x) = x^5 + \lfloor 2x \rfloor$ and let μ_F and μ_G be the Lebesgue-Stieltjes measures they induce on the interval $(0,10)$. Find the Lebesgue-Radon-Nikodym decomposition of μ_F with respect to μ_G.

6. Let X be a Banach space and $T \in L(X)$ a linear operator with $\|T\| < 1$. Prove that $I - T$ is invertible. (Hint. Find a series that converges to $(I-T)^{-1}$.)

7. Give an example of a function f with the property that $f \in L^p((2,\infty))$ if and only if $p \geq 2$.

8. Find, with proof, a constant C such that

$$\left(\int_0^2 f(x)dx\right)^5 \leq C \int_0^2 f^5(x)dx$$

for all functions f that are positive and continuous on the interval $[0,2]$.

9. Find the Fourier transform of the one variable function $f(x) = x^3 e^{-2\pi x^2}$.

10. Prove that for every $0 < x < 2$ the following formula is valid

$$\frac{x}{2} = \frac{\pi}{2} - \frac{\sin x}{1} - \frac{\sin 2x}{2} - \frac{\sin 3x}{3} - \cdots.$$