Real Analysis Preliminary Examination

2003

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded.

1. Consider the set $X = \{a, b, c\}$. Starting with the map ϕ given by

$$\phi(\emptyset) = 0, \quad \phi(\{a\}) = 1,$$
$$\phi(\{a,b\}) = 2, \quad \phi(\{b\}) = 3,$$
$$\phi(\{a,b,c\}) = 5,$$

   and using the Carathéodory extension, construct an outer measure μ^* on X. Is the set $\{a, b\}$ μ^*-measurable?

2. Let $F(x) = 3x + \lfloor \sqrt{x} \rfloor$, where $\lfloor x \rfloor$ is the greatest integer function of x. Denote by μ_F the Lebesgue-Stieltjes measure associated with F. Compute the integral

$$\int_{[0,9]} 2^x \, d\mu_F.$$

3. Prove the formula

$$\frac{1}{a^3} = \frac{1}{2} \int_0^\infty x^2 e^{-ax} dx.$$

   Using this formula, rigorously derive

$$\sum_{n=1}^\infty \frac{1}{n^3} = \frac{1}{2} \int_0^\infty \frac{x^2}{e^x - 1} dx$$

4. Show that

$$\int_0^\infty e^{-sx} x^{-1} \sin^2(x) \, dx = \frac{1}{4} \ln(1 + 4s^{-2}), \quad s > 0$$

   by integrating $e^{-sx} \sin(2xy)$ on $[0, \infty) \times [0, 1]$. Be sure that you verify the hypothesis of the theorems that you use (Hint: A half-angle formula might be useful.)

5. Let $F, G, H : [0, \infty) \to \mathbf{R}$, $F(x) = x + \lfloor x \rfloor$, $G(x) = \lfloor 2x \rfloor$, $H(x) = x^3$, where $\lfloor x \rfloor$ is the greatest integer function, and let μ_F, μ_G and μ_H be the Lebesgue-Stieltjes measures they determine. Compute the Radon-Nikodym derivatives

$$\frac{d\mu_F}{d\mu_G}, \quad \frac{d\mu_G}{d\mu_H}, \quad \frac{d\mu_H}{d\mu_F}$$

   if they exist. If not, explain why they do not exist.

6. Give an example of a function that is in $L^5(\mathbf{R})$ but not in $L^7(\mathbf{R})$. Give an example of a function that is in $L^7(\mathbf{R})$ but not in $L^5(\mathbf{R})$.

7. Prove that in any Hilbert space, the following polarization formula holds

$$< x, y >= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right).$$

8. Let $K(x,t) \in L^2(\mathbf{R^2})$. Define

$$(Tf)(x) = \int_{\mathbf{R}} K(x,t) f(t) dt.$$

Prove that T is a bounded linear operator from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$.

9. Let $f(x) = \frac{1}{2} - x$ on the interval $[0, 1)$, and extend f to be periodic on $\mathbf{R}$.
a. Find the Fourier series of f.
b. Deduce the formula

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

10. Prove that for any real number y,

$$\sum_{m=-\infty}^{\infty} e^{-2\pi(m+y)^2} = \frac{\sqrt{2}}{2} \sum_{m=-\infty}^{\infty} e^{-\pi m^2/2} e^{2\pi i m y}.$$