Real Analysis Preliminary Examination

August, 2004

Do 7 of the following 11 problems. You must clearly indicate which 7 are to be graded.
Notation: $\mathbb{R}$ = the set of real numbers. m = Lebesgue measure.

1. A subset E of $\mathbb{R}$ is called isolated if every $x \in E$ has a neighborhood N such that $E \cap N = \{x\}$. Prove that any isolated subset of $\mathbb{R}$ is countable.

2. Prove that if μ^* is an outer measure on X and $\{A_j\}_1^\infty$ is a sequence of disjoint measurable sets, then

$$\mu^*(E \cap (\bigcup_{j=1}^{\infty} A_j)) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j)$$

for any $E \in X$.

3. Let μ_F be the Lebesgue-Stieltjes measure associated to a nondecreasing continuous function $F : \mathbb{R} \to \mathbb{R}$. Prove that if $E \subset \mathbb{R}$ is countable, then it is μ_F-measurable, and $\mu_F(E) = 0$.

4. Let $f(x)$ be a Lebesgue integrable function on $\mathbb{R}$. Prove that the function

$$F(x) = \int_{-\infty}^{x} f \; dm$$

is continuous.

5. Let $C \subset [0, 1]$ be the Cantor ternary set. Define $f(x) : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x & \text{if } x \notin C \\ 0 & \text{if } x \in C. \end{cases} .$$

Prove that f is Lebesgue measurable.

6. Let $|f_n| \leq g$ and $f_n \to f$ in measure. Prove that $|f| \leq g$ a.e. with respect to that measure.

7. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ-finite measure spaces, and $f(x, y)$ be a nonnegative function on $X \times Y$ such that

   i. for each fixed $x \in X$, $f(x, y)$ is a ν-measurable function on Y, and
   ii. the function $F(x) = \int_Y f(x, y) \; d\nu(y)$ is not μ-measurable.

   Prove that $f(x, y)$ is not $(\mu \times \nu)$-measurable.

8. Prove that any bounded variation function $F(x)$ on $[a, b]$ can be written as $F(x) = G(x) + H(x)$ where $G(x)$ is absolutely continuous and $H'(x) = 0$ a.e. on $[a, b]$ (with respect to the Lebesgue measure).

9. **a.** Prove the Parallelogram Law:

$$\|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2)$$

   for any f, g in a Hilbert space $\mathcal{H}$, where $\|f\| = \sqrt{\langle f, f\rangle}$ is the induced norm.

   b. Use **a.** to prove that $L^p(\mathbb{R}, m)$ is a Hilbert space if and only if $p = 2$.

10. Let $p, q > 1$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Prove that for any $g \in L^q(\mathbb{R}, m)$ the functional $G : L^p(\mathbb{R}, m) \to \mathbb{R}$ defined by

$$G(f) = \int_{\mathbb{R}} f(x)g(x) \; dm$$

is a well defined bounded linear functional, and $\|G\| = \|g\|_q$.

11. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed vector spaces and $T : \mathcal{X} \to \mathcal{Y}$ be a bounded linear map. Prove that

$$T^{-1}(\{0\}) := \{x \in \mathcal{X} : T(x) = 0\}$$

is a closed subspace of $\mathcal{X}$.