Real Analysis Doctoral Preliminary Examination
May 2004

Solve 7 out of 10 problems given below. Clearly indicate the problems to be graded.

1. **a)** Give an example (with proof) of a sequence of functions $f_n : [a,b] \to \mathbb{R}$, $(n = 1, 2, \dots)$, a and b are finite, and a function $f : [a,b] \to \mathbb{R}$ with the following properties:

(i) $f_n \to f$ pointwise on $[a,b]$ as $n \to \infty$;

(ii) $|f_n(x)| \le c < \infty$ for all $n \in \mathbb{N}$ and all $x \in [a,b]$;

(iii) $f_n \in \mathcal{R}[a,b]$ for all $n \in \mathbb{N}$, where $\mathcal{R}[a,b]$ is the class of all Riemann integrable functions on $[a,b]$;

(iv) $\lim\limits_{n\to\infty} \int_a^b f_n \, dx$ exists;

(v) $f \notin \mathcal{R}[a,b]$

b) Prove that for any sequence $\{f_n\}$ satisfying (i)-(v) in a), f is Lebesgue integrable and

$$\lim_{n\to\infty} \int_a^b f_n \, dx = \int_a^b f \, dx \tag{1}$$

if the integral on the right in (1) is understood as a Lebesgue integral. (You can refer to known convergence theorems from the theory of measure and integration.)

2. Let $(X, \mathcal{M})$ be a measurable space and $\mu : \mathcal{M} \to [0, \infty]$ be a finitely additive set function. Assume that μ is continuous from below, i.e.,

$$\text{if } E_j \in \mathcal{M} \text{ for each } j \text{ and } E_1 \subset E_2 \subset \dots, \text{ then } \mu(\bigcup_{j=1}^{\infty} E_j) = \lim_{j\to\infty} \mu(E_j).$$

Prove that μ is a countable additive measure.

3. Let X be a set and $\mathcal{P}(X)$ be the collection of all subsets of X. Assume that $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra and $\mu_0 : \mathcal{A} \to [0, \infty]$ is a premeasure on $\mathcal{A}$. For any $E \subset X$ define

$$\mu^*(E) = \inf\{\sum_{j=1}^{\infty} \mu_0(A_j) : A_j \in \mathcal{A}, E \subset \cup_{j=1}^{\infty} A_j\}.$$

Accept without proof that μ^* is an outer measure on X. Prove that

a. $\mu^*(E) = \mu_0(E)$ for any $E \in \mathcal{A}$;

b. if $E \in \mathcal{A}$ then E is μ^*-measurable in the sense of the Carathéodory definition.

4. Let $(X, \mathcal{M})$ be a measurable space and $\{f_j\}_{j=1}^{\infty}$ be a sequence of $[-\infty, \infty]$-valued $\mathcal{M}$-measurable functions on X. Prove that if $\lim\limits_{n\to\infty} f_n(x)$ exists for all $x \in X$ and $f(x)$ is the function defined by $f(x) = \lim\limits_{n\to\infty} f_n(x)$, then f is also an $\mathcal{M}$-measurable function on X. (Hint: first prove that the functions g_1, g_2, g_3, g_4 are all $\mathcal{M}$-measurable where

$$g_1(x) = \sup_j f_j(x), \quad g_2(x) = \inf_j f_j(x), \quad g_3(x) = \limsup_{j\to\infty} f_j(x), \quad g_4(x) = \liminf_{j\to\infty} f_j(x).)$$

5. Let

i) $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$;

ii) $f : X \to [-\infty, \infty]$ be a measurable function such that $|f(x)| \le c$ for a finite $c > 0$ and for a.e. $x \in X$.

Consider a sequence of subdivisions σ_n $(n = 1, 2, \dots)$ of the interval $[-c, c]$:

$$-c = t_n^0 < t_n^1 < \cdots < t_n^{n-1} < t_n^n = c$$

with $\delta_n = \max\limits_{0 \le k \le n-1} (t_n^{k+1} - t_n^k) \to 0$ as $n \to \infty$, and define

$$E_n^k = f^{-1}([t_n^k, t_n^{k+1})) \text{ for } 0 \le k \le n-2 \text{ and } E_n^{n-1} = f^{-1}([t_n^{n-1}, t_n^n]).$$

Pick $\xi_n^k \in [t_n^k, t_n^{k+1}]$ for every $n \in \mathbb{N}, k = 0, \dots, n-1$. Prove that

i)

$$f \in L^1(X, \mathcal{M}, \mu),$$

ii)

$$\int_X f d\mu = \lim_{n \to \infty} \sum_{k=0}^{n-1} \xi_n^k \mu(E_n^k),$$

i.e., the integral of f is equal to the limit of the Lebesque integral sums.

(You can refer to known convergence theorems from the theory of integration.)

6. Let $(X, \mathcal{M}, \mu)$ be a measure space, f_n $(n = 1, 2, \dots)$, f are measurable functions on X, and $f_n \to f$ almost uniformly, i.e., for any $\varepsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \varepsilon$ and $\sup\limits_{x \in E^c} |f_n(x) - f(x)| \to 0$ as $n \to \infty$. Prove that

a) $f_n \to f$ in measure;

b) $f_n \to f$ a.e.

7. Let $F : [a, b] \to \mathbb{R}$ be an increasing function (i.e. if $x < y$ then $F(x) \le F(y)$). Accept the fact without proof that the derivative $F'(x)$ exists for a.e. $x \in [a, b]$ in Lebesgue measure.

a) Prove that the function F' (which is defined a.e.) is Lebesgue measurable and

$$\int_a^b F'(x) dx \le F(b) - F(a).$$

(Hint: define $F_n(x) = n(F(x + \frac{1}{n}) - F(x))$, where F is extended outside $[a, b]$ by the rule $F(x) = F(b)$ for $x > b$. Notice that $F_n \to F'$ a.e.)

b) Describe (without proof) an example of F for which the strict inequality holds.

8. Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \leq p < \infty$.

i) Give the definition of the space $L^p = L^p(X, \mathcal{M}, \mu)$.

ii) Accept the fact without proof that the Minkowsky's inequality holds, i.e., L^p is a normed vector space. Prove that L^p is complete as a metric space, i.e., L^p is a Banach space.

(Hints:

a) you can use without proof the criterion of completeness of normed metric spaces in terms of absolutely convergent series of vectors;

b) you can refer without proofs to appropriate convergence theorems from the theory of integration.)

9. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ with the inner product denoted by (x, y) for $x, y \in \mathcal{H}$ and the induced norm $\|x\| = \sqrt{(x, x)}$. Let $\{\varphi_k\}_{k=1}^{\infty} \subset \mathcal{H}$ be an orthonormal system of vectors.

a) Prove the Bessel's inequality:

$$\sum_{k=1}^{\infty} |(x, \varphi_k)|^2 \leq \|x\|^2 \text{ for any } x \in \mathcal{H}.$$

b) Prove that $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal basis in $\mathcal{H}$ if and only if the Parseval's equality holds:

$$\sum_{k=1}^{\infty} |(x, \varphi_k)|^2 = \|x\|^2 \text{ for any } x \in \mathcal{H}.$$

10. Let $X = Y = [0, 1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$ be the Borel σ-algebra, μ be the Lebesgue measure on $[0, 1]$, and ν be the counting measure on $[0, 1]$, i.e., $\nu(\{x\}) = 1$ for any $x \in [0, 1]$. Consider the measure spaces $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$, and $(X \times Y, \mathcal{M} \times \mathcal{N}, \mu \times \nu)$. Let $\mathcal{D} = \{(x, x) \in X \times Y : x \in [0, 1]\}$ be the diagonal of the square $X \times Y$ and

$$\mathcal{X}_{\mathcal{D}}(x, y) = \begin{cases} 0 \text{ if } x \neq y \\ 1 \text{ if } x = y \end{cases}$$

i) Prove that the integrals

$$\int_Y \int_X \mathcal{X}_{\mathcal{D}}(x, y) d\mu(x) d\nu(y), \quad \int_X \int_Y \mathcal{X}_{\mathcal{D}}(x, y) d\nu(y) d\mu(x), \quad \int_{X \times Y} \mathcal{X}_{\mathcal{D}} d(\mu \times \nu)$$

exist, but are all unequal. (Hint: use the definition of $\mu \times \nu$ to compute the last integral.)

ii) Which of the conditions of the Fubini-Tonelli theorem is not satisfied in this example?