Real Analysis Preliminary Examination

August, 2005

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded.
Notation: $\mathbb{R}$ = the set of all real numbers; $\mathbb{Q}$ = the set of all rational numbers; m = Lebesgue measure; $\widetilde{E}$ = the complement of E.

1. Let $\mathbb{N}$ be the set of all integers, $\mathbb{N}^+$ be the set of all positive integers, and $\mathbb{N}^-$ be the set of all negative integers. Starting with the map μ given by

$$\mu(\emptyset) = 0, \quad \mu(\{x\}) = 1 \text{ for all } x \in \mathbb{N}, \quad \mu(\mathbb{N}^+) = 1, \quad \mu(\mathbb{N}^-) = 1$$

and using the Carathèodory extension, construct an outer measure μ^* on $\mathbb{N}$. Find all μ^*-measurable sets.

2. Let $E_0 = [0,1]$, and E_n, $n = 1, 2, \ldots$, be the set obtained from E_{n-1} by removing middle open interval of length $1/5^n$ from each of the closed intervals of E_{n-1}. Let $E = \bigcap_{n=0}^{\infty} E_n$. Find $m(E)$ and show that E is a closed set not containing any nonempty open intervals.

3. Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series (i.e. $\sum_{n=1}^{\infty} |a_n|$ converges). Prove that the sum of series is independent of the order of summation; i.e. prove that for any finite subsets $E_1 \subset E_2 \subset E_3 \subset \cdots \subset \mathbb{N}$ such that $\bigcup_{n=1}^{\infty} E_n = \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, we have

$$\lim_{n\to\infty} \sum_{k\in E_n} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

4. Let $\{r_1, r_2, \ldots\}$ be a numeration of all rational numbers in $[0,1]$, and define a function F on $\mathbb{R}$ by

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ \displaystyle\sum_{r_n \le x} \frac{1}{2^n} & \text{if } x \ge 0. \end{cases}$$

Accept the fact without proof that F is nondecreasing, right-continuous. Let μ_F be the Lebesgue-Stieltjes measure defined by F, and

$$\phi(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Compute

$$\int_{[0,1]} \phi(x)\, d\mu_F$$

and justify your answer.

5. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two σ-finite measure spaces, and f be a $\mu \times \nu$-measurable function on $X \times Y$ such that for almost all fixed y, the function $x \mapsto f(x,y)$ is an integrable function on X, and the function $y \mapsto \int_X |f(x,y)|\, d\mu$ is an integrable function on Y. Prove that

$$\int_{X\times Y} f\, d\mu \times \nu = \int_Y \left(\int_X f\, d\mu \right) d\nu = \int_X \left(\int_Y f\, d\nu \right) d\mu.$$

6. Give an example of sequence of functions $\{f_n\}$ defined on $[0,1]$ such that $f_n \to 0$ in Lebesgue measure on $[0,1]$, but $f_n(x) \not\to 0$ for any $x \in [0,1]$.

7. Let $(X, \mathcal{M})$ be a measurable space, and μ be a finite measure on $\mathcal{M}$. Prove that if $r > s \ge 1$ then

$$L^r(X,\mu) \subset L^s(X,\mu).$$

8. Assume f is integrable (with respect to Lebesgue measure) on $[a,b]$ and

$$\int_a^c f(t)\, dt = 0$$

for any $c \in [a,b]$. Prove that $f = 0$ a.e. on $[a,b]$.

9. Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$, F be a nondecreasing, right-continuous function on $\mathbb{R}$, and μ_F be the Lebesgue-Stieltjes measure defined by F. Prove that $\mu_F \ll m$ on $(\mathbb{R}, \mathcal{B})$ if and only if F is absolutely continuous.

10. Let A be a one-to-one bounded linear operator from a Banach space X to a Banach space Y and $S \subset Y$ be the range of A. Prove that $A^{-1} : S \mapsto X$ is bounded if and only if S is closed in Y.