Real Analysis Preliminary Examination

May, 2005

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded.
Notation: $\mathbb{R}$ = the set of all real numbers; $\mathbb{Q}$ = the set of all rational numbers; m = Lebesgue measure; $\widetilde{E}$ = the complement of E.

1. Let $\mathcal{M} = \{E \subset \mathbb{R} : \text{either } E \text{ is countable or } \widetilde{E} \text{ is countable.}\}$ and μ be the measure on $\mathcal{M}$ defined by

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is countable,} \\ 1 & \text{if } \widetilde{E} \text{ is countable.} \end{cases}$$

Prove that a function $f(x)$ defined on $\mathbb{R}$ is μ-measurable if and only if there is a constant c such that $f(x) = c$ for all but countably many x-values in $\mathbb{R}$.

2. Let f be an integrable function on a measure space $(X, \mathcal{M}, \mu)$. Prove that if

$$\int_E f \, d\mu \leq 0 \text{ for all } E \in \mathcal{M}$$

then $f \leq 0$ μ-a.e.

3. A measure on the Borel σ-algebra $\mathcal{B}$ is called a Baire measure if it is finite for bounded sets. Prove that μ is a Baire measure on $\mathbb{R}$ if and only if it is a Lebesgue-Stieltjes measure defined by a nondecreasing, right-continuous function F.

4. Give an example of measure spaces $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$, and a non-negative measurable function on $X \times Y$ such that

$$\int_Y \left(\int_X f \, d\mu \right) d\nu \neq \int_X \left(\int_Y f \, d\nu \right) d\mu.$$

5. Compute $\displaystyle\lim_{n\to\infty} \int_0^\infty \frac{x}{1+x^n} \, dx$. Justify your steps.

6. Let $f, g \in L^1(\mathbb{R})$. Show that

$$h(x) = \int_{-\infty}^{\infty} f(y) g(x-y) \, dy$$

is in $L^1(\mathbb{R})$. Justify your steps.

7. Let $x = \sum_{n=1}^\infty \frac{a_n}{3^n}, a_n \in \{0, 1, 2\}$ be a ternary expansion of $x \in [0, 1]$, and f be the Cantor ternary function on $[0, 1]$ defined by

$$f\left(\sum_{n=1}^\infty \frac{a_n}{3^n} \right) = \sum_{n=1}^{N-1} \frac{a_n/2}{2^n} + \frac{a_N}{2^N}$$

where

$$N = \begin{cases} \infty, & \text{if } a_n \neq 1 \text{ for all } n, \\ \min\{n : a_n = 1\}, & \text{otherwise.} \end{cases}$$

You may assume that f is well defined. Prove that f is a function of bounded variation on $[0, 1]$, but not absolutely continuous.

8. Let ν be a signed measure on a measurable space $(X, \mathcal{B})$. Prove that the Hahn decomposition $\nu = \nu^+ - \nu^-$ is unique.

9. Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Prove that for any bounded linear operator $A : L^p[a, b] \to L^p[a, b]$, there is a unique bounded linear operator $A^* : L^q[a, b] \to L^q[a, b]$ with $\|A^*\| \leq \|A\|$ such that

$$\int_a^b A(f) g \, dm = \int_a^b f A^*(g) \, dm$$

for all $f \in L^p[a, b]$ and $g \in L^q[a, b]$.

10. Let H be an inner product space over $\mathbb{R}$. Prove that $\{x_1, x_2, \ldots x_n\} \subset H$ is an orthogonal system if and only if

$$\|a_1 x_1 + \cdots + a_n x_n\|^2 = \|a_1 x_1\|^2 + \cdots + \|a_n x_n\|^2$$

for any $a_1, \ldots, a_n \in \mathbb{R}$.