# Real Analysis Preliminary Exam
August 2006

**Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded.**

**1.** Let $\mu$ be a finite measure on $\mathbb{R}$ absolutely continuous with respect to Lebesgue measure, and $g \in L^\infty(\mu)$. Show that the function

$$G(t) = \int_{\mathbb{R}} g\chi_{[0,t]}\, \mathrm{d}\mu$$

is continuous on $\mathbb{R}$. (Recall that for $a, b \in \mathbb{R}$, $[a, b] = \{x \in \mathbb{R} \mid a \le x \le b\}$.)

**2.** Give the definition of Lebesgue integral and use it to show that if $f$ is Lebesgue integrable on $[a, b]$ and $c > 0$ and $d \in \mathbb{R}$ then

$$\int_a^b f(x)\, \mathrm{d}x = c \int_{\frac{a-d}{c}}^{\frac{b-d}{c}} f(cx + d)\, \mathrm{d}x.$$

**3.** If $f \in L^1[0, 1]$ and $g \in L^\infty[0, 1]$, show that $fg \in L^1[0, 1]$ and $\|fg\|_1 \le \|f\|_1 \|g\|_\infty$. Give an example for which equality holds, and an example for which strict inequality holds.

**4.** a. Prove the 'Parallelogram Law' for Hilbert spaces:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

b. Show that the Banach space $C[0, 1]$ with the norm $\|f\| = \sup_{x\in[0,1]} |f(x)|$ does not admit an inner product.

**5.** Suppose $f$ is Lebesgue integrable on $[0, 1]$.

a. If $f$ is continuous on $[0, 1]$ and $\int_0^1 |f|\, \mathrm{d}x = 0$, show that $f$ is identically zero on $[0, 1]$.

b. Show that the previous statement is false without the continuity assumption.

c. Define precisely $L^1[0, 1]$ and show that $\|g\|_1 = \int_0^1 |g|\, \mathrm{d}x$ defines a norm on $L^1[0, 1]$ (be sure to resolve the seeming contradiction with Part b!).

**6.** Show that a continuous real-valued function on $(0, 1)$ is the uniform limit of a sequence of polynomials on $(0, 1)$ if and only if $f$ can be extended to a continuous function on $[0, 1]$.

**7.** Let $m^*$ denote Lebesgue outer measure on $\mathbb{R}^2$. If $p(x, y)$ is a nonzero polynomial and $Z = \{(x, y) \in \mathbb{R}^2 \mid p(x, y) = 0\}$, show that $m^*Z = 0$.

**8.** Prove that the Radon-Nikodym derivative, when it is defined, is unique modulo sets of measure zero.

**9.** Let $\mathcal{B}$ denote the Borel subsets of $\mathbb{R}$ and $m$ denote Lebesgue measure on $\mathcal{B}$. For $E \in \mathcal{B}$ define

$$\nu E = \sum_{n=0}^{\infty} \frac{m(E \cap [n, n+1))}{2^n}.$$

Show that $(\mathbb{R}, \mathcal{B}, \nu)$ is a finite measure space and find the Radon-Nikodym derivative $\left[\frac{\mathrm{d}\nu}{\mathrm{d}m}\right]$.

**10.** Prove or disprove by counterexample: If $g(t)$ is continuous on $[0, 1]$, then there is a subinterval $(a, b) \subset [0, 1]$ with $a < b$ for which $g$ is monotonic on $(a, b)$.