Real Analysis Preliminary Exam
May 2006

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded.

1. Let

$$g(x) = \sum_{n=1}^{\infty} \frac{\chi_{[n,n+1)}(x)}{n^2},$$

and for Lebesgue measurable sets E define $\nu E = \int_E g(x)\,\mathrm{d}x$. Find $\int_{\mathbb{R}} 1\,\mathrm{d}\nu$ and $\int_{\mathbb{R}} x\,\mathrm{d}\nu$.

2. Suppose $f(x)$ is Lebesgue measurable on $[0,1]$. Show that $g(x,y) = f(x) - f(y)$ is measurable on $[0,1] \times [0,1]$ with respect to the two-dimensional Lebesgue measure.

3. Let f be a continuous real-valued function on the unit interval $[0,1]$. Show that for each $\epsilon > 0$ there exists a nonnegative integer n and $c_0, \ldots, c_n \in \mathbb{R}$ so that

$$\left|c_0 + c_1 e^{-x} + c_2 e^{-2x} + \cdots + c_n e^{-nx} - f(x)\right| < \epsilon \qquad \text{for all } x \in [0,1].$$

4. Consider the function $h(x)$ defined by

$$h(x) = \begin{cases} -x^2, & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{Z} \setminus \{0\} \\ x^2, & \text{otherwise.} \end{cases}$$

Is $h(x)$ of bounded variation on $[0,1]$? (Prove your answer, of course!)

5. Let $(X, \mathcal{B}, \mu)$ be a finite measure space with $\mu X = 1$. If $E_1, E_2, \ldots, E_{16}$ are measurable sets with $\mu E_j = 1/3$ for each j, show that for some $1 \le j_1 < j_2 < j_3 < j_4 < j_5 < j_6 \le 16$, $\mu(E_{j_1} \cap \cdots \cap E_{j_6}) > 0$.

6. For $f, g \in L^1(\mathbb{R})$ the convolution $f * g$ is defined by $(f * g)(x) = \int_{\mathbb{R}} f(x-t)g(t)\,\mathrm{d}t$. For $f \in L^1(\mathbb{R})$, the *Fourier transform* $\hat{f}$ of f is defined by $\hat{f}(s) = \int e^{ist} f(t)\,\mathrm{d}t$. Show that $\hat{f}$ is a bounded complex function and

$$\widehat{f * g} = \hat{f}\hat{g}.$$

(Recall: If F_1, F_2 are integrable real-valued functions, the integral of the complex-valued function $F = F_1 + iF_2$ is $\int F = \int F_1 + i \int F_2$.)

7. (Riemann-Lebesgue Theorem) If f is integrable on $\mathbb{R}$, show that

$$\lim_{k \to \infty} \int_{\mathbb{R}} f(x) \cos(kx)\,\mathrm{d}x = 0.$$

8. Find

$$\lim_{n \to \infty} \int_a^{\infty} \frac{n}{1 + n^2 x^2}\,\mathrm{d}x$$

for $a > 0, a = 0$ and $a < 0$. Justify all of your steps!

9. State and prove Fatou's Lemma and give an example in which strict inequality occurs.

10. Suppose f and its derivative f' are absolutely continuous on $[0,1]$ with f' increasing. Set $g(x,y) = f''(x+y)\chi_{[0,1]}(x+y)$ and show that

$$\int_{[0,1]\times[0,1]} g(x,y)\,\mathrm{d}(x \times y) = f'(1) + f(0) - f(1).$$

(Justify your steps).