Real Analysis Preliminary Examination

August, 2007

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded. Strive for clear and detailed solutions.

1. For a fixed $x_0 \in \mathbb{R}$, let $\mu(E) = \begin{cases} 1. & \text{if } x_0 \in E \\ 0. & \text{if } x_0 \notin E \end{cases}$.

   (a) Prove that μ is a measure on the σ-algebra $\mathcal{P}(\mathbb{R})$ of all subsets of $\mathbb{R}$.

   (b) Find $\displaystyle\int f d\mu$ for a given function f defined on $\mathbb{R}$.

2. Prove or disprove: Every real-valued measurable function g on $\mathbb{R}$ which satisfies

$$m(\{x \in \mathbb{R} : |g(x)| > \alpha\}) < \frac{1}{\alpha^2}, \quad \forall \alpha \in (0, \infty), \text{ is integrable on } \mathbb{R}.$$

3. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{f_k\}_{k=1}^{\infty}$ be a sequence in $L^p(X, \mathcal{M}, \mu)$ such that $\sum_{k=1}^{\infty} \|f_k\|_p < \infty$. Prove that $\sum_{k=1}^{\infty} f_k$ converges in the L^p norm.

4. Suppose $f \in L^p(X, \mathcal{M}, \mu)$ for all $1 \leq p < \infty$ and that there exists a constant $c \geq 0$ such that $\|f\|_p \leq c$ for all $1 \leq p < \infty$. Prove that $f \in L^\infty(X, \mathcal{M}, \mu)$.

5. Let ν be a signed measure on the measurable space $(X, \mathcal{M})$ and let $|\nu| = \nu^+ + \nu^-$, where $\nu = \nu^+ - \nu^-$ is the Jordan Decomposition of ν. Prove that $|\nu|(E) = \sup \sum_{i=1}^{n} |\nu(E_i)|$, where the supremum is taken over all $\bigcup_{i=1}^{n} E_i = E$ with E_i pairwise disjoint.

6. Define a Borel measure μ on a topological space to be *regular* if each Borel set B has the following property: For each $\epsilon > 0$, there exists an open set U and a closed set F such that $F \subset B \subset U$ and $\mu(U \setminus F) < \epsilon$. Prove that if μ is a finite Borel measure on a metric space X, then μ is regular. (*Hint: Consider the class $\mathcal{A}$ of Borel sets which have the stated regularity property.*)

7. Let $g : [a.b] \to \mathbb{R}$ be absolutely continuous and monotone. Prove that if $E \subset [a, b]$ with $m(E) = 0$ then $m(g(E)) = 0$.

8. Suppose that ν is a σ-finite signed measure and μ is a σ-finite measures on $(X, \mathcal{M})$ such that $\nu \ll \mu$. If $g \in L^1(X, \mathcal{M}, \nu)$, prove that

$$g\frac{d\nu}{d\mu} \in L^1(X, \mathcal{M}, \mu) \text{ and } \int g \, d\nu = \int g \frac{d\nu}{d\mu} \, d\mu.$$

9. Let $T : H \to H$ be a bounded linear operator on a Hilbert space H. Prove that

   (a) there exists a unique bounded linear operator T^* on H such that $\langle Tf, g\rangle = \langle f, T^*g\rangle$ for all $f, g \in H$ and

   (b) $\|T\| = \|T^*\|$.

   (*Hint: for (b), first prove $\|T\| = \sup\{|\langle Tf, g\rangle| : \|f\| \leq 1, \|g\| \leq 1\}$.*)

10. Let $f \in L^1((0, a))$ and $g(x) = \displaystyle\int_x^a \frac{f(t)}{t} \, dt$ for $0 < x < a$. Prove that $g \in L^1((0, a))$ and

$$\int_0^a g(x) \, dx = \int_0^a f(x) \, dx.$$