Real Analysis Preliminary Examination

May, 2007

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded. Strive for clear and detailed solutions.

1. Let $0 \le f_n \le f$, where f_n and f are measurable functions and $\lim_{n\to\infty} f_n(x) = f(x)$ for every x. Prove that $\lim_{n\to\infty} \int f_n(x) = \int f(x)$.

2. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Show that there are at most countably many $a \in X$ for which $\mu(\{a\}) > 0$.

3. a. Let $B(X)$ denote the space of bounded complex-valued functions on the set X with the uniform metric $\rho(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$. Prove that $B(X)$ is complete.

   b. If X is a topological space, let $BC(X)$ denote the space of bounded continuous complex-valued functions on X. Prove that $BC(X)$ is a closed subspace of $B(X)$.

4. Let $f \in L^2(\mathbb{R})$ with respect to Lebesgue measure. Prove that f can be represented in the form $f = g + h$, where $g \in L^1(\mathbb{R})$ and $h \in L^\infty(\mathbb{R})$.

5. Let $P = (0, \infty)$. Define a Borel measure μ on P by $\mu(B) = \int_B \frac{dx}{x}$. Let $f, g \in L^1(\mu)$.

   a. Prove that the set
   $$A = \{x \in P : t \mapsto f(x/t)g(t) \text{ is in } L^1(\mu)\}$$
   satisfies $\mu(P \setminus A) = 0$.

   b. Prove that the function
   $$h(x) = \begin{cases} \int_0^\infty f(x/t)g(t)\, d\mu(t), & x \in A \\ 0, & \text{otherwise} \end{cases}$$
   is in $L^1(\mu)$.
   (Note: You need not prove that $f(x/t)$ is measurable.)

6. Let X and Y be Banach spaces. Let $\{T_n\}$ be a sequence of bounded linear operators from X to Y such that for each $x \in X$, $\lim_{n\to\infty} T_n(x)$ exists in Y. Define $T(x) = \lim_{n\to\infty} T_n(x)$. Prove that T is a bounded linear operator from X to Y.

7. A function $f : [a, b] \to \mathbb{R}$ is called Lipschitz if there exists $M > 0$ such that $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in [a, b]$. Prove that f is Lipschitz if and only if it is absolutely continuous and $f'(x)$ is bounded a.e. on $[a, b]$.

8. a. State the Radon-Nikodym Theorem.

   b. Prove it for σ-finite measures, assuming the result for finite measures.
   (Note: you need not prove uniqueness.)

9. Let $f : [a, b] \to \mathbb{R}$ be a Lebesgue integrable function, and let $\epsilon > 0$. Prove that there exists a polynomial p such that $\int_a^b |f(x) - p(x)|\, dx < \epsilon$.

10. Let f be a real valued function on $(0, 1)$ and $M > 0$.

    a. Suppose that f is differentiable on $(0, 1)$ and $|f'| \le M$. Prove that f is bounded.

    b. Suppose that f is differentiable a.e. (w.r.t. Lebesgue measure) on $(0, 1)$ and $|f'| \le M$ everywhere that f' is defined. Does it follow that f is bounded? Fully justify your answer.