Real Analysis Preliminary Examination

August, 2008

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded. Strive for clear and detailed solutions.

1. Let S be a collection of subsets of $\mathbb{R}$ consisting of $\mathbb{R}$ and all finite subsets of $\mathbb{R}$. Define a set function μ on S: $\mu(E) = 1$ if $E = \mathbb{R}$ and $\mu(E) = 0$ if E is finite.
   (a) Construct an outer measure μ^* on $\mathbb{R}$ using Carathéodory extension.
   (b) Determine the σ-algebra of all μ^*-measurable sets.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space, and f, g be μ-measurable functions. Prove $f + g$ is also μ-measurable.

3. Let $(X, \mathcal{M})$ be a measure space and $\mu : \mathcal{M} \to [0, \infty)$ be a finitely additive set function. Assume that
$$\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j) \quad \text{for all measurable } E_1 \supset E_2 \supset \cdots$$
Prove that μ is a measure.

4. Let μ^* be an outer measure on a set X, and $E_1, E_2 \subset X$ be μ^*-measurable sets. Prove that
$$\mu^*(E_1 \cap E_2 \cap A) + \mu^*((E_1 \cup E_2) \cap A) = \mu^*(E_1 \cap A) + \mu^*(E_2 \cap A)$$
for any subset $A \subset X$.

5. Prove that a function with continuous first derivative on $[a, b]$, $-\infty < a < b < \infty$, is a function of bounded variation on $[a, b]$.

6. Prove that for any integrable function f, the set $S = \{x \in X : f(x) \neq 0\}$ is σ-finite.

7. (a) Prove that $\lim_{x\to\infty} \int_x^{\infty} f\, dx = 0$ for any Lebesgue integrable function on $\mathbb{R}$.
   (b) Give an example of Lebesgue measurable function f such that $\lim_{x\to\infty} \int_x^{\infty} f\, dx \neq 0$

8. Let $F(x) = \chi_{[0,\infty)}$, μ_F^* be the Lebesgue-Stieltjes outer measure generated by F, and μ_F be the corresponding Lebesgue-Stieltjes measure. Determine the σ-algebra of all μ_F^*-measurable sets and compute $\int_{\mathbb{R}} e^{x^2}\, d\mu_F$.

9. Let X be a normed vector space and $\{x_1, \ldots, x_n\} \subset X$ be a set of linearly independent vectors. Prove that there are bounded linear functionals $\{f_1, \ldots, f_n\}$ such that
$$f_i(x_j) = \begin{cases} 0, & \text{if } i \neq j \\ \inf_{y \in \text{span}\{x_l,\ l \neq i\}} \{\|y - x_i\|\}, & \text{if } i = j \end{cases}$$

10. Let $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $x \in \mathbb{R}^n$ and $1 \leq p < \infty$. Accept the fact that $(\mathbb{R}^n, \|\cdot\|_p)$ is a Banach space. Prove that there are $0 < c_1 < c_2 < \infty$ such that
$$c_1\|x\|_1 \leq \|x\|_p \leq c_2\|x\|_1 \quad \text{for all } x \in \mathbb{R}^n.$$
(*Hint: Open Mapping Theorem*)