Real Analysis Preliminary Examination

May, 2008

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded. Strive for clear and detailed solutions.

1. Let $\mathcal{S} = \{E_1, E_2, \ldots\}$ be a disjoint collection of subsets of $\mathbb{R}$. Find the σ-algebra generated by $\mathcal{S}$.

2. Let $F(x)$ be a nondecreasing, absolutely continuous function on $\mathbb{R}$, μ_F be the Lebesgue-Stieltjes measure generated by F, and m be the Lebegue measure on $\mathbb{R}$. Prove $\mu_F \ll m$ and $\left[\frac{d\mu_F}{dm}\right] = F'$.

3. Consider the measure space $(\mathbb{R}, \mathcal{M}, \mu)$ where $\mathcal{M}$ consists of all countable subsets of $\mathbb{R}$ and their complements, $\mu(E) = 0$ if E is countable, and $\mu(E) = 1$ if the complement of E is countable. Let $\mathbb{Q} = \{r_1, r_2, \ldots\}$ be an enumeration of the rational numbers. Define $f(x) = 2^n$ if $x = r_n$, and $f(x) = 1$ otherwise.

   (a) Prove that $f(x)$ is μ-measurable.

   (b) Compute $\int_{\mathbb{R}} f \, d\mu$.

4. Let f be a Lebesgue integrable function on $\mathbb{R}$. Prove that if $\int_O f \, dm \geq 0$ for any open set $O \subset \mathbb{R}$, then $f \geq 0$ m-a.e.

5. Prove that a monotone function $f(x)$ on $[a, b]$ is absolutely continuous if and only if $\int_a^b f'(x) \, dx = f(b) - f(a)$.

6. Let $1 \leq p < q < r < \infty$, and $0 < k < 1$ be the unique number satisfies $\frac{1}{q} = \frac{k}{p} + \frac{1-k}{r}$. Prove that $\|f\|_q \leq (\|f\|_p)^k (\|f\|_r)^{1-k}$ for any measurable function f and hence $L^p \cap L^r \subset L^q$.
   (*Hint:* $q = kq + (1-k)q$)

7. Let f be an integrable function and $E_n = \{x : |f(x)| > n\}$. Prove that
$$\lim_{n\to\infty} \int_{E_n} |f| \, d\mu = 0.$$

8. Let $f, g \geq 0$ be two Lebesgue measurable functions on $\mathbb{R}$. Define $f * g(x) = \int_{\mathbb{R}} f(x-t)g(t) \, dt$. Prove
$$\int_{\mathbb{R}} f * g(x) \, dx = \left(\int_{\mathbb{R}} f(x) \, dx\right)\left(\int_{\mathbb{R}} g(x) \, dx\right).$$
   Justify your steps.

9. Let X be a normed vector space. Prove that for any $x \in X$, there is a bounded linear functional f on X such that $f(x) = \|x\|$ and $\|f\| = 1$.

10. Let $C[a, b]$ be the set of all continuous real valued functions on $[a, b]$. Let $g \in C[a, b]$ be a monotone function. Prove that
$$\mathcal{A} = \{a_0 + a_1 g + a_2 g^2 + \cdots + a_n g^n : a_i \in \mathbb{R}, n = 0, 1, \ldots\}$$
   is dense in $C[a, b]$ under the L^∞ norm.