Real Analysis Preliminary Examination August, 2009

Do 7 of the following 9 problems. You must clearly indicate which 7 are to be graded. Strive for clear and detailed solutions.

1. Let $(H, \langle\ ,\ \rangle)$ be a Hilbert space and $A : H \to H$ be a bounded linear operator. Show that there is a unique bounded linear operator $A^* : H \to H$ such that $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x, y \in H$. Show that $\|A^*\| = \|A\|$.

2. Prove the following theorem (Egoroff's Theorem):

   Theorem: Let (X, μ) be a measure space with $\mu(X) < \infty$. Suppose that f, f_n, $n = 1, 2, \ldots$ are measurable functions such that $f_n \to f$ a.e. Then for every $\epsilon > 0$ there exists a measurable set $E \subset X$ such that $\mu(E) < \epsilon$ and such that $f_n \to f$ uniformly on $E^c = X \backslash E$.

3. Let m be Lebesgue measure on $\mathbb{R}$. Show that for $f \in L^p(m)$, $p \geq 1$, we have that $f_h \in L^p(m)$ where $f_h(x) := f(x+h)$. Show that $\lim_{h\to 0} \|f - f_h\| = 0$.

4. Show that if $(f_n)_{n=1}^{\infty}$ is a sequence of measurable extended real valued functions on a measure space X, then the set $\{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is measurable.

5. For $t \in \mathbb{R}$, let $[t]$ denote the greatest integer not exceeding t. Let $F(t) := t + [t]$. Find
$$\int_0^{\infty} e^{-t}\, dF.$$

6. State and prove the Closed Graph Theorem

7. Compute $\lim_{n\to\infty} \int_0^{\infty} \frac{x}{1+x^n}\, dx$. Justify all steps.

8. Give an example of measure spaces (X, M, μ), (Y, N, ν), and a non-negative measurable function f defined on $X \times Y$ such that
$$\int_Y \left(\int_X f(x,y)\, d\mu(x) \right) d\nu(y) \neq \int_X \left(\int_Y f(x,y)\, d\nu(y) \right) d\mu(x).$$

9. State and prove the Monotone Convegence Theorem for general measure spaces (use Fatou's Lemma).