Real Analysis Preliminary Examination May, 2009

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded. Strive for clear and detailed solutions.

1. Let f be a nonnegative function in $L^1(m)$ where m is the Lebesgue measure on the real line $\mathbb{R}$. Show that the function g defined by
$$g(x) := \int_{(-\infty,x)} f\,dm$$
is a continuous function.

2. Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be normed vector spaces. Define what it means for a linear map from $X \to Y$ to be bounded. Let $L(X, Y)$ be the set of bounded linear maps from X to Y. Define the operator norm on $L(X, Y)$ and show that if $(Y, \|\cdot\|_2)$ is a Banach space then $L(X, Y)$ is a Banach space (with the operator norm).

3. Let $\{u_\alpha\}_{\alpha\in A}$ be an indexed orthonormal set in a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ where $\langle\cdot,\cdot\rangle$ is conjugate linear in the second slot. Show that the following are equivalent:
a) If $\langle x, u_\alpha\rangle = 0$ for all $\alpha \in A$ then $x = 0$.
b) For each $x \in \mathcal{H}$, $x = \sum_{\alpha\in A}\langle x, u_\alpha\rangle u_\alpha$ where this sum has only countably many nonzero terms and the convergence is in the norm derived from $\langle\cdot,\cdot\rangle$ and independent of the order of the nonzero terms.
(You may use Bessel's Inequality.)

4. Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose that g is a non-negative measurable function on X. Show that
a) $\nu(E) := \int_E g\,d\mu$ for $E \in \mathcal{M}$ defines a measure on X (with domain $\mathcal{M}$).
b) for f a non-negative measurable function on X we have $\int_X f\,d\nu = \int_X f\,g\,d\mu$.

5. Suppose that $f : [0,1]\times[0,1] \to \mathbb{R}$ satisfies the following conditions:
a) f is bounded;
b) For each x, the map $t \mapsto f(x,t)$ is measurable;
c) The partial derivative $\frac{\partial f}{\partial t}$ exists everywhere and is bounded.
Show that
$$\frac{d}{dt}\int_0^1 f(x,t)\,dx = \int_0^1 \frac{\partial f}{\partial t}(x,t)\,dx$$
where the integral is the Lebesgue integral.

6. Let X be a Banach space. Show that
$$\|x\| = \sup\{|\phi(x)| : \phi \in X^* \text{ with } \|\phi\| = 1\}.$$

7. Let $g : [a,b] \to \mathbb{R}$ be absolutely continuous. Show that if $E \subset [a,b]$ has Lebesgue measure zero then $g(E)$ also has Lebesgue measure zero.

8. Prove the Riemann-Lebesgue lemma: If $f \in L^1(\mathbb{R})$ then
$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\cos(nx)\,dx = 0$$
(Here we are using Lebesgue integration on the line).

9. Let m be the Lebesgue measure on the real line $\mathbb{R}$. Show by example that there exists a sequence of functions in $L^1(m)$ that converges to zero in $L^1(m)$ but such that the sequence does *not* converge to zero pointwise almost everywhere.

10. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. Suppose that $K \in L^2(X\times Y, \mu\otimes\nu)$. Show that if $f \in L^2(Y, \nu)$ then the formula
$$(Tf)(x) := \int K(x,y)f(y)\,d\nu(y)$$
defines a function $Tf \in L^2(X, \mu)$. Show that $T : L^2(X, \mu) \to L^2(Y, \nu)$ is a bounded linear operator satisfying $\|Tf\|_2 \le \|K\|_2\,\|f\|_2$.