# Real Analysis Preliminary Examination

August, 2010

**Do 7 of the following 9 problems. You must clearly indicate which 7 are to be graded. Strive for clear and detailed solutions.**

1. Let $\mu$ be a complete measure. Prove that if $f$ is a measurable function and if $f = g$ $\mu$-a.e., then $g$ is a measurable function.

2. Let $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\emptyset \in \mathcal{E}$, $X \in \mathcal{E}$, and $\rho(\emptyset) = 0$. $\forall A \subset X$, define

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \rho(E_j) \,\middle|\, E_j \in \mathcal{E} \text{ and } A \subset \bigcup_{j=1}^{\infty} E_j \right\}.$$

   Prove that $\mu^*$ is an outer measure.

3. Let $f$ be a nonnegative element of $L^1[0,1]$. Prove that

$$\lim_{n\to\infty} \int_0^1 (f(x))^{1/n}\, dx = m(\{x \in [0,1] | f(x) > 0\}).$$

4. Suppose $\{f_n\} \subset L^+$ (nonnegative measurable functions), $f_n \to f$ pointwise, and $\int f = \lim_{n\to\infty} f_n < \infty$. Prove that for all $E \in \mathcal{M}$

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

5. Let $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(\mathbb{R})$, and $g \in L^q(\mathbb{R})$. Prove that

$$f * g(x) = \int_{\mathbb{R}} f(x-y) g(y)\, dy$$

   exists for every $x$, $||f * g||_\infty \le ||f||_p ||g||_q$, and $f * g$ is uniformly continuous.

6. Suppose $f$ is absolutely continuous on $\mathbb{R}$ and $f \in L^1(\mathbb{R})$. Prove that if, in addition,

$$\lim_{t\to 0^+} \int_{\mathbb{R}} \left| \frac{f(x+t) - f(x)}{t} \right| dx = 0,$$

   then $f \equiv 0$. (*Hint: begin your work with Fatou*)

7. Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable. Assuming that Lebesgue measure is translation invariant, prove that the Lebesgue integral is translation invariant, i.e. prove that if $f \in L^1(\mathbb{R}^n)$, then

$$\int f(x)\, dm^n = \int f(x+y)\, dm^n.$$

8. Suppose that there exists a $p < \infty$ such that $f \in L^q \cap L^\infty$ for all $q \ge p$. Prove that

$$||f||_\infty = \lim_{q\to\infty} ||f||_q.$$

   (*Warning: Your argument should also show that the limit exists.*)

9. Let $X$ and $Y$ be normer vector spaces and $T$ be a bounded linear transformation from $X$ to $Y$. Define $S : Y^* \to X^*$ by $S(f) = f \circ T$. Prove that $S$ is a bounded linear transformation and that $||S|| = ||T||$. (*Possible hint: Hahn-Banach.*)