15

Real Analysis Preliminary Examination

May, 2010

Do 7 of the following 9 problems. You must clearly indicate which 7 are to be graded. Strive for clear and detailed solutions.

1. Suppose $f, g : X \to [-\infty, \infty]$ are measurable. Prove that fg is measurable (where $0 \cdot (\pm\infty) = 0$).

2. Let $f \in L^1(\mathbb{R}^1)$. Prove that $\lim_{n\to\infty} \int_{-\infty}^{\infty} (\cos x)^n f(x)\, dx = 0$.

3. Let μ and λ be positive measures on a σ-algebra $\mathcal{A}$. Prove that the following two statements are equivalent:

   a) $\forall A \in \mathcal{A}$, if $\mu(A) = 0$, then $\lambda(A) = 0$.

   b) $\forall \epsilon > 0$, $\exists \delta > 0$, such that $\forall A \in \mathcal{A}$, if $\mu(A) < \delta$, then $\lambda(A) < \epsilon$.

   (*Possible hint for one direction: Proceed by contradiction.*)

4. Prove that if f, g are complex-valued functions in $L^1(X, \mathcal{M}, \mu)$, then

$$\int_E f = \int_E g \ \ \forall E \in \mathcal{M} \text{ iff } f = g \text{ a.e.}$$

5. Suppose $1 \le p < \infty$. If $f_n, f \in L^p$ and if $f_n \to f$ a.e., then prove that

$$\|f_n - f\|_p \to 0 \text{ iff } \|f_n\|_p \to \|f\|_p.$$

6. Prove that if $f \in L^1(0,1)$ and $a > 0$, then the integral

$$F_a(x) = \int_0^x (x-y)^{a-1} f(y)\, dy$$

   exists for a.e. $x \in (0,1)$ and $F_a \in L^1(0,1)$.

7. Prove that $L^\infty(X, \mathcal{M}, \mu)$ is complete.

8. Prove that $C_c(\mathbb{R}^n)$ (continuous functions with compact support in $\mathbb{R}^n$) is dense in $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$.

9. Let H be an infinite-dimensional Hilbert space. Prove that every orthonormal sequence in H converges weakly to 0.

Dw