Real Analysis Preliminary Examination

August 2011

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded. Strive for clear and detailed solutions.

1. Let $E \subseteq \mathbb{R}$ be Lebesgue measurable and $1 \leq p < \infty$. Suppose $\{f_n\} \to f$ in $L^p(E)$. Show that there is a subsequence $\{f_{n_k}\}$ and a function $g \in L^p(E)$ such that for all $k \geq 1$, $|f_{n_k}| \leq g$ a.e. on E.

2. Let ψ be an integrable simple function on $\mathbb{R}$ and $\epsilon > 0$. Show that there is a step function φ on $\mathbb{R}$ with $\int_{\mathbb{R}} |\psi - \varphi| \, \mathrm{d}m < \epsilon$. (This follows from a density result proven in class, but you may not cite the result - you are being asked to prove this here).

3. Let X and Y be normed vector spaces and $T : X \longrightarrow Y$ linear. Show that T is a bounded linear transformation if and only if T is continuous.

4. Let $\mathcal{S} = 2^{\mathbb{R}}$ be the powerset of $\mathbb{R}$ and $\mathbb{N} = \{1, 2, \ldots\}$. Define $\mu : \mathcal{S} \longrightarrow [0, \infty)$ by

$$\mu A = \begin{cases} \inf(A \cap \mathbb{N}), & \text{if } A \cap \mathbb{N} \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

(i) Determine the Carathéodory measure $\overline{\mu}$ on $\mathbb{R}$ induced by μ.
(ii) Is $\overline{\mu}$ an extension of μ?

5. (Chebychev's Inequality) Let $(X, \mathcal{B}, \mu)$ be a measure space, f a nonnegative measurable function on X and $\lambda > 0$. Show that $\mu\left(f^{-1}\left([\lambda, \infty]\right)\right) \leq \frac{1}{\lambda} \int_X f \, \mathrm{d}\mu$.

6. Show that if $\{f_n\}$ is a sequence of measurable functions on the measurable space $(X, \mathcal{B})$ then $\underline{\lim} f_n$ is a measurable function on $(X, \mathcal{B})$.

7. Let $\mathcal{B}$ denote the collection of Lebesgue measurable subsets of $\mathbb{R}$. For each $E \in \mathcal{B}$ define

$$\mu E = \sum_{n \geq 1} \frac{1}{n^3} \int_{E \cap [n, n+1)} x \, \mathrm{d}x,$$

Is $(\mathbb{R}, \mathcal{B}, \mu)$ a σ-finite measure space? Is it a finite measure space? Is μ absolutely continuous with respect to Lebesgue measure m? Find the Radon-Nikodym derivative $\left[\frac{\mathrm{d}\mu}{\mathrm{d}m}\right]$ or show that it does not exist.

8. Show that if X is a Banach space then so is X^* with respect to the (usual) operator norm. (We proved a more general result in class, which you *may not cite* - you should prove this result here. You may assume that X^* is a vector space, but should prove that X^* has every other property required of a Banach space).

9. Let m denote Lebesgue measure on $\mathbb{R}$ and for all $E \subseteq \mathbb{R}$ define μE to be the cardinality of $E \cap \mathbb{Z}$. Find $\int_{[2,\infty)\times[2,\infty)} (y-1)x^{-y}\, d(m(x) \times \mu(y))$. If you invoke a theorem call it by name and be sure to explicitly verify that all of the hypotheses are satisfied.

10. Show that if $f \in C[0,1]$ and $x_0 \in (0,1)$ then the limit

$$\lim_{n\to\infty} \frac{n}{2} \int_{(x_0-\frac{1}{n},x_0+\frac{1}{n})} f \, dm,$$

exists. Does the conclusion still hold if the hypothesis is relaxed to $f \in L^\infty[0,1]$?