Real Analysis Preliminary Examination

May 2011

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded. Strive for clear and detailed solutions.

1. Let m denote the Lebesgue measure on $\mathbb{R}$. Prove or disprove: there is a measurable set $A \subseteq \mathbb{R}$ with the property that for every bounded interval I, $m(A \cap I)/mI = 1/2$.

2. Let $(X, \mathcal{B})$ be a measurable space and $x_0 \in X$. Define a measure δ_{x_0} on this space by $\delta_{x_0} E = \chi_E(x_0)$ (you may assume without proof that δ_{x_0} is a measure). Show that if f is δ_{x_0}-measurable then $\int_X f \, d\delta_{x_0} = f(x_0)$.

3. Let $(X, \mathcal{B}, \mu)$ be a measure space and $\{f_n\}$ a sequence of measurable functions on X. Let E denote the set of points x at which the sequence $\{f_n(x)\}$ does *not* converge. Show that E is measurable.

4. Suppose f is a Lebesgue integrable function on $\mathbb{R}$ with the property that $\int_I f \, dm = 0$ for every interval I. Show that $f = 0$ a.e..

5. Suppose $f \in C[a, b]$ and for every nonnegative integer k, $\int_{[a,b]} f(x) e^{kx} \, dx = 0$. Show that $f \equiv 0$ on $[a, b]$.

6. Let $(X, \mathcal{B}, \mu)$ be a finite measure space.
(i) Show that $L^2(X, \mu) \subseteq L^1(X, \mu)$.
(ii) Suppose that $\{f_n\}_{n \geq 1}$ is a Cauchy sequence of functions in $L^2(X, \mu)$. Show that $\{f_n\}_{n \geq 1}$ is also Cauchy with respect to the L^1-norm.

7. Determine $\displaystyle\lim_{n \to \infty} \int_{[0,\infty)} \frac{1}{x^{1/n} + x^n} \, dx$. (As always, you must *justify* each step.)

8. Show that $L^\infty[0, 1]$ is not separable.

9. Let H be a Hilbert space with inner product $\langle -, - \rangle$ and u a nonzero element of H. Define a function F on H by $F(x) = \langle u, x \rangle$.
(i) Show that F is continuous on H.
(ii) Show that F is an open mapping.

10. Let $X = Y = \{1, 2, \ldots\}$ and $\mu = \nu$ be the counting measure on X and Y respectively. Let

$$f(x, y) = \begin{cases} 2 - 2^{-x}, & \text{if } x = y, \\ -2 + 2^{-x}, & \text{if } x = y + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Compute $\int_X \int_Y f(x, y) \, d\nu(y) \, d\mu(x)$, $\int_Y \int_X f(x, y) \, d\mu(x) \, d\nu(y)$, $\int_{X \times Y} f^+(x, y) \, d(\mu(x) \times \nu(y))$ and $\int_{X \times Y} f^-(x, y) \, d(\mu(x) \times \nu(y))$. Explain in detail why these results don't contradict the theorems of Fubini and Tonelli.