Real Analysis Preliminary Examination

August, 2012

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded. Strive for clear and detailed solutions.

1. Let μ^* be an outer measure on X. A collection of subsets $\{E_1, E_2, \ldots\}$ of X is called a partition of X if $E_i \cap E_j = \emptyset$, for any $i \neq j$, and $\bigcup_{i=1}^{\infty} E_i = X$. Prove that all the subsets in a partition $\{E_n\}_1^\infty$ are μ^*-measurable if and only if $\mu^*(A) = \sum_{i=1}^{\infty} \mu^*(A \cap E_i)$ for any subset A of X.

2. Let S be a dense subset of $\mathbb{R}$ and $(X.\mathcal{M})$ be a measurable space. Prove that a real valued function f is measurable if and only if $\{x : f(x) \leq r\} \in \mathcal{M}$ for all $r \in S$.

3. Let E be a Lebesgue measurable subset of $\mathbb{R}$ with $m(E) < \infty$. Prove that for any $\epsilon > 0$, there is a finite disjoint union of open intervals A such that $m(E \bigtriangleup A) = m((E \setminus A) \cup (A \setminus E)) < \epsilon$.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space, and $\{f_n, n = 1, 2, \ldots\}$ be a sequence of measurable functions which converges a.e. Prove that if there exists a subsequence $\{n_i\}$ such that $\lim_{i \to \infty} \int |f_{n_i}| \, d\mu = 0$, then $\lim_{n \to \infty} f_n(x) = 0$ a.e.

5. Let $\{f_n\}$ be a sequence of measurable functions over a σ-finite measure space $(X, \mathcal{M}, \mu)$. Prove that if $\sum_{n=1}^{\infty} |f_n|$ is integrable, then each f_n is integrable, $\sum_{n=1}^{\infty} f_n$ converges almost everywhere and is integrable, and
$$\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu.$$

6. Let $-\infty < a < b < \infty$ and f be a function of bounded variation on $[a, b]$. Prove that f can be written as $f = g + h$, where g is absolutely continuous and $h' = 0$ a.e. on $[a, b]$.

7. Let $(X, \mathcal{M})$ be a measurable space, ν be a signed measure on $(X, \mathcal{M})$, $X = P \cup N$ be a Hahn decomposition for ν. For any $E \in \mathcal{M}$, prove that
$$|\nu(E)| = |\nu|(E) \quad \text{if and only if either } |\nu|(E \cap P) = 0 \text{ or } |\nu|(E \cap N) = 0.$$

8. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space X over $\mathbb{R}$. Prove that if there is a $c > 0$ such that $\|x\|_1 \leq c\|x\|_2$ for all $x \in X$, and X is complete with respect to both norms, then the two norms are equivalent.

9. Let H be an infinite dimensional Hilbert space. Prove that the unit sphere $S = \{f \in H : \|f\| = 1\}$ contains a sequence that converges to 0 weakly (Simply using the fact that S is weakly dense in the unit ball is not acceptable).

10. Let $(X, \mathcal{M}, \mu)$ be a measure space, and $f \in L^p \cap L^\infty$ for some $1 \leq p < \infty$ with $\|f\|_\infty > 0$. For any $0 < \alpha < 1$, let
$$E_\alpha = \{x : |f(x)| > \alpha \|f\|_\infty\}.$$
Prove that
$$0 < \mu(E_\alpha) < \infty, \quad \text{and} \quad \|f\|_p \geq \alpha \|f\|_\infty \mu(E_\alpha)^{1/p}.$$