Real Analysis Preliminary Examination

May, 2012

Do 7 of the following 10 problems. You must clearly indicate which 7 are to be graded. Strive for clear and detailed solutions.

1. Prove that an outer measure μ^* on a σ-algebra $\mathcal{M}$ is a measure if and only if
$$\mu^*(E \cup F) = \mu^*(E) + \mu^*(F),$$
for any disjoint sets E and F in $\mathcal{M}$.

2. Let $(X, \mathcal{M})$ be a measurable space. Prove that for any sequence of measurable functions $\{f_1, f_2, \ldots\}$, $\sup_n f_n$ is a measurable function.

3. Let F be an increasing, right continuous function on $\mathbb{R}$, and μ_F be the Lebesgue-Stieltjes measure associated to F. Prove that
$$\mu_F(\{a\}) = F(a) - F(a^-).$$

4. Let $f, f_n, n = 1, 2, \ldots$ be in $L^P(X, \mathcal{M}, \mu)$ for a p satisfying $1 \leq p < \infty$, $|f_n| \leq |f|$ for all n, and $\lim_{n\to\infty} f_n = f$ a.e. Prove that $f_n \to f$ in L^p.

5. Let $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and f be a $\mathcal{M} \otimes \mathcal{N}$-measurable function on $X \times Y$. Prove that if
$$\int\int |f(x,y)| \, d\mu(x) d\nu(y) < \infty$$
then f is $\mu \times \nu$-integrable and
$$\int f \, d(\mu \times \nu) = \int\int f(x,y) \, d\mu(x) d\nu(y) = \int\int f(x,y) \, d\nu(y) d\mu(x).$$

6. Let $-\infty < a < b < \infty$. Prove that every function of bounded variation on $[a, b]$ is Riemann integrable on $[a, b]$.

7. Let $(X, \mathcal{M})$ be a measurable space, μ be a positive measure and ν be a signed measure on $(X, \mathcal{M})$. Prove that if $|\nu(E)| \leq \mu(E)$ for all $E \in \mathcal{M}$, then
$$\nu \ll \mu \quad \text{and} \quad \left|\frac{d\nu}{d\mu}\right| \leq 1 \ \ \mu\text{-a.e.}$$

8. Let $-\infty < a < b < \infty$ and f be a continuous function on $[a, b]$. Prove that if
$$\int_a^b x^n f(x) \, dx = 0 \ \text{ for all } n = 0, 1, 2, \ldots, \text{ then } \ f \equiv 0 \ \text{ on } \ [a, b].$$

9. Let $(X, \mathcal{M}, \mu)$ be a measure space over $\mathbb{R}$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and $g \in L^q$. Define $\phi : L^p \to \mathbb{R}$:
$$\phi(f) = \int fg \, d\mu.$$
Prove that ϕ is a bounded linear functional on L^p and $\|\phi\| = \|g\|_q$.

10. Prove that if $f \in L^p(X, \mathcal{M}, \mu)$ then $E = \{x : f(x) \neq 0\}$ is σ-finite.