Real Analysis Preliminary Exam
August 2013

Directions: Complete exactly seven (7) of the following ten problems, and indicate in the boxes below which seven problems should be graded.

If you do not indicate exactly seven problems in the boxes above, up to seven worked problems will be graded in the order they appear below. Strive for clear, concise, and legible solutions. If a reader cannot easily follow your argument you may not receive credit for that problem. If any problems are nearly identical to an unnamed textbook result, you should assume that you're being asked to prove that result.
Unless otherwise indicated, m denotes the Lebesgue measure.

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1. Let m^2 denote the Lebesgue measure on $\mathbb{R}^2$ and let $P = [0, \infty) \times [0, \infty)$. Compute

$$\lim_{n\to\infty} \int_P \frac{e^{-x-y}}{1 + (x+y)^{1/n}} \, \mathrm{d}m^2.$$

(As usual, you must justify all steps!)

2. (i) State the Radon-Nikodym (or Lebesgue-Radon-Nikodym) Theorem.
(ii) Let $\mathcal{L}$ be the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$. Let m denote the Lebesgue measure on $(\mathbb{R}, \mathcal{L})$ and ν the counting measure on $(\mathbb{R}, \mathcal{L})$. Find $\frac{\mathrm{d}m}{\mathrm{d}\nu}$ or prove that it does not exist. Why does this example not contradict the Radon-Nikodym Theorem?

3. Let μ^* be an outer measure on X and $\mathcal{M}$ the σ-algebra of μ^*-measurable sets. Suppose $F \subset X$ has the property that for all $\epsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu^*(E \triangle F) < \epsilon$. Show that $F \in \mathcal{M}$.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \longrightarrow \mathbb{C}$ a μ-measurable function. If $\int |f| \, \mathrm{d}\mu = 0$ then $f = 0$ almost everywhere.

5. Show that if $f \in L^1(\mathbb{R}, m)$ then for all $\epsilon > 0$ there exists $\delta > 0$ such that for every measurable set A with $mA < \delta$, $\int_A |f| \, \mathrm{d}m < \epsilon$.
Hint: use the fact that the simple functions are dense in $L^1(\mathbb{R}, m)$.

6. Suppose $0 < p < q < \infty$. Show that $L^p(\mathbb{R}, m) \not\subset L^q(\mathbb{R}, m)$ and $L^q(\mathbb{R}, m) \not\subset L^p(\mathbb{R}, m)$.

7. Let X, Y be Banach spaces, $L(X, Y)$ the space of bounded linear functions from X to Y and $\{T_n\}$ a sequence in $L(X, Y)$ which converges pointwise on X to a function T. Show that $T \in L(X, Y)$.

8. Show that for every bounded Lebesgue measurable set $E \subset \mathbb{R}$, $mE = \sup\{mK : K \text{ is compact and } K \subset E\}$.

9. Let $\{u_\alpha\}_{\alpha \in A}$ be an orthonormal basis for a Hilbert space H. Show that for all $x, y \in H$,

$$\langle x, y \rangle = \sum_{\alpha \in A} \langle x, u_\alpha \rangle \overline{\langle y, u_\alpha \rangle}.$$

10. Let H be a Hilbert space. Show that H has a countable orthonormal basis if and only if H is separable.