Real Analysis Preliminary Exam
May 2013

Directions: Complete exactly seven (7) of the following ten problems, and indicate in the boxes below which seven problems should be graded.

If you do not indicate exactly seven problems in the boxes above, up to seven worked problems will be graded in the order they appear below. Strive for clear, concise, and legible solutions. If a reader cannot easily follow your argument you may not receive credit for that problem. If any problems are nearly identical to an unnamed textbook result, you should assume that you're being asked to prove that result.
Unless otherwise indicated, m denotes the Lebesgue measure.

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1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Show that the set $B(X, \mu)$ of all bounded complex-valued functions in $L^1(X, \mu)$ is dense in $L^1(X, \mu)$. *Hint:* prove it first for real-valued functions.

2. Let E be a closed subspace of a Hilbert space H and $\{v_\alpha\}_{\alpha \in A}$ an orthonormal basis for E. For each $x \in H$ define $p_E(x) = \sum_{\alpha \in A} \langle x, v_\alpha \rangle v_\alpha$.
(i) Show that for each $x \in H$, the sum defining $p_E(x)$ converges and for each $\alpha_0 \in A$, $\langle p_E(x), v_{\alpha_0} \rangle = \langle x, v_{\alpha_0} \rangle$.
(ii) For each $x \in H$, $p_E(x)$ is the unique vector in E closest to x.

3. Let $1 \leq p < \infty$, and let S be the closed unit sphere in $L^p(\mathbb{R}, m)$: $S = \{f \in L^p(\mathbb{R}, m) : \|f\|_p = 1\}$.
(i) Construct a sequence $\{f_n\} \subset S$ such that $\|f_i - f_j\|_p \geq 1$ for all $i \neq j$.
(ii) Show that S is not compact (despite being closed and bounded) by exhibiting an open cover of S which has no finite subcover.

4. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is Lebesgue measurable and for every Lebesgue measurable set E, $\left|\int_E f \, \mathrm{d}m\right| \leq mE$. Show that $\|f\|_\infty \leq 1$.

5. Suppose ν is a signed measure on $(X, \mathcal{M})$ and f is a measurable real-valued function on X such that $\int_E f \, \mathrm{d}\nu = 0$ for all $E \in \mathcal{M}$. Show that $f = 0$ except possibly on a null set.

6. Let E be the set of all real numbers in $[0, 1)$ which have a decimal (base-10) expansion containing a 5. Show that E is Lebesgue measurable and find mE.

7. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuously differentiable and f' is bounded on $\mathbb{R}$. Show that f is absolutely continuous.

8. Let $D = \{(x, x) : x \in [0, 1]\} \subset \mathbb{R}^2$. Let m denote the Lebesgue measure on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ and ν the counting measure on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ and assume without proof that D is $m \times \nu$ measurable.

(i) Use the definition of product measure to compute $(m \times \nu)D$.
(ii) Find $\int \chi_{_D} \, \mathrm{d}(m \times \nu)$, $\iint \chi_{_D} \, \mathrm{d}m \, \mathrm{d}\nu$, and $\iint \chi_{_D} \, \mathrm{d}\nu \, \mathrm{d}m$.
(iii) Explain why the results of Part (ii) do not contradict Fubini's Theorem. Explain why the results of Part (ii) do not contradict Tonelli's Theorem.

9. Let μ and ν be σ-finite measures on $(X, \mathcal{M})$ with $\nu \ll \mu$ and let f be a nonnegative measurable function on X. Show that $\int f \, \mathrm{d}\nu = \int f \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \, \mathrm{d}\mu$.
Hint: Show it first for simple functions.

10. For each positive integer n, let $f_n(t) = \frac{t}{1+t^n}$. Compute $\lim_{n\to\infty} \int_0^\infty f_n(t) \, \mathrm{d}t$. As usual, be sure to justify all steps!