Real Analysis **Preliminary Examination** August 2014

Do 7 of the 9 problems. You must clearly indicate which problems are to be graded. If you don't do this, then problems 1 - 7 will be graded. Strive for clear and detailed solutions.

On this exam $(X, \mathcal{M}, \mu)$ denotes a general measure space. $\overline{\mathbb{R}}$ denotes the extended real numbers $[-\infty, \infty]$.

1. Define $\phi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ such that
$$\phi(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$
(Assume that $\frac{1}{\pm\infty} = 0$.) Prove that ϕ is Borel measurable.

2. Let (f_n) be a sequence of $\overline{\mathbb{R}}$-valued measurable functions on a set $D \in \mathcal{M}$. Suppose that for all $\eta > 0$ there exists an $\mathcal{M}$-measurable subset $E \subset D$ such that $\mu(E) < \eta$ and $\lim_{n\to\infty} f_n(x)$ exists for all $x \in D \setminus E$. Prove that $\lim_{n\to\infty} f_n(x)$ exists a.e. on D.

3. Let $f \in L^1(X)$ and let $\epsilon > 0$. Prove that there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and
$$\left| \int_X f - \int_E f \right| < \epsilon.$$

4. Using the Fubini/Tonelli theorems to justify all steps, evaluate
$$\int_0^1 \int_y^1 x^{-3/2} \cos\left(\frac{\pi y}{2x}\right) dx\, dy$$

5. Let ν be a signed measure. Define $|\nu|$ as $\nu^+ + \nu^-$, where $\nu = \nu^+ - \nu^-$ is the Jordan decomposition of ν. Prove that
$$|\nu|(E) = \sup_{(E_i)} \left\{ \sum_{i=1}^n |\nu(E_i)| : E_i \text{ are disjoint and } E = \bigcup_{i=1}^n E_i \right\}.$$

6. Let $f : \mathbb{R} \to \mathbb{R}$, $f \in L^1(\mathbb{R})$. Let
$$g(x) = \int_{\mathbb{R}} e^{-|y|} f(x-y)\, dy, \quad x \in \mathbb{R}.$$
   (a) Prove that g is continuous on $\mathbb{R}$.
   (b) Prove that g is of bounded variation on $\mathbb{R}$. (Possible hint: Let $h(x) = e^{-|x|}$. Express h in terms of $\psi(x) = e^{\min(0,x)}$.)

7. If μ and ν are finite positive measures on $(X, \mathcal{M})$, prove there exists a nonnegative measurable function f on X such that for all $E \in \mathcal{M}$,
$$\int_E (1-f)\, d\mu = \int_E f\, d\nu.$$

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8. Let H be an infinite-dimensional Hilbert space.
   (a) Prove that every orthonormal sequence in H converges weakly to 0.
   (b) Prove that the unit sphere $S = \{x : ||x|| = 1\}$ is weakly dense in the unit ball $B = \{x : ||x|| \leq 1\}$.

9. Suppose $0 < \mu(X) < \infty$ and $f \in L^p(X)$ for all $1 \leq p < \infty$ and suppose there exists a constant $C > 0$ such that $||f||_p \leq C$ for all p, $1 \leq p < \infty$. Prove that $f \in L^\infty(X)$. (Possible hint: Guess what $||f||_\infty$ might be and construct a proof by contradiction.)