Real Analysis **Preliminary Examination** May 2014

Do 7 of the 9 problems. You must clearly indicate which problems are to be graded. If you don't do this, then problems 1 - 7 will be graded. Strive for clear and detailed solutions.

On this exam, $(X, \mathcal{M}, \mu)$ denotes a general measure space (except in problem 7.) m_n denotes Lebesgue measure on $\mathbb{R}^n$. $\mathcal{L}$ denotes the σ-algebra of Lebesgue measurable sets on $\mathbb{R}^1$. $\overline{\mathbb{R}}$ denotes the extended real numbers $[-\infty, \infty]$.

1. Let $f, g : X \to \overline{\mathbb{R}}$ be measurable functions. Define

$$h(x) = \begin{cases} 0, & \text{if } f(x) = -g(x) = \pm\infty \\ f(x) + g(x) & \text{otherwise.} \end{cases}$$

   Prove that h is measurable.

2. (a) For all $E \in \mathcal{P}(\mathbb{R})$ (the set of all subsets of $\mathbb{R}$) and for all $\alpha \in \mathbb{R}$, prove that $m^*(\alpha E) = |\alpha|\, m^*(E)$, where m^* denotes Lebesgue outer measure.

   (b) For all $E \in \mathcal{L}$ and for all $\alpha \in \mathbb{R}$, prove that $\alpha E \in \mathcal{L}$.

3. Evaluate

$$\lim_{n\to\infty} \int_0^\infty \sum_{k=1}^{n} \frac{(-1)^k x^{2k}}{(2k)!} e^{-2x}\, dx,$$

   and justify your answer. (Possible hint: $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$.)

4. Let $\mu(X) < \infty$ and let $f, f_1, f_2, \ldots$ be real-valued measurable functions on X. Prove that if $f_n \to f$ a.e., then $f_n \to f$ in measure. (Possible hint: Consider an intersection of unions.)

5. Let μ and ν be measures on $(X, \mathcal{M})$ such that $\mu \geq \nu$.

   Define $\lambda(E) = \sup\{\mu(F) - \nu(F) : F \in \mathcal{M}, F \subset E, \text{ and } \nu(F) < \infty\}$. Prove that λ is a measure on $\mathcal{M}$. (Hint: First show λ is finitely additive.)

6. Suppose that $f : [a, b] \to \mathbb{R}$ is increasing. Prove that

$$\int_a^b f'(t)\, dt \leq f(b) - f(a).$$

   You may assume that f' exists and is nonnegative a.e. on $[a, b]$.

7. Let X be a normed vector space and let x_0 be a nonzero vector in X. Prove there exists a bounded linear functional $F : X \to \mathbb{R}$ such that $||F|| = 1$ and $F(x_0) = ||x_0||$.

8. Let A be an open subset of $\mathbb{R}^2$. For $h > 0$, define $B_h = \{(x, y, h) \in \mathbb{R}^3 : (x, y) \in A\}$. Define $C = \{(\lambda x, \lambda y, \lambda z) : (x, y, z) \in B_h, 0 \leq \lambda \leq 1\}$. Show that $m_3(C) = \frac{1}{3} h m_2(A)$. You need not prove that C is measurable.

9. Let $1 < p < \infty$. Suppose (f_n) is a sequence of functions in $L^p([0,1])$ and suppose that for all n, $||f_n||_p \leq 1$ and that $f_n(x) \to 0$ a.e. Prove that for all $g \in L^q([0,1])$, where $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_0^1 f_n g \, dm_1 \to 0.$$

(Hint: Egoroff)