Real Analysis **Preliminary Examination** August 2015

Do 7 of the following 9 problems. You must clearly indicate which problems are to be graded. If you do not do this, then problems 1-7 will be graded. Strive for clear and detailed solutions. $(X, \mathcal{M}, \mu)$ denotes a measure space. m_n denotes Lebesgue measure on $\mathbb{R}^n$. m denotes Lebesgue measure on $\mathbb{R}$. $\overline{\mathbb{R}}$ denotes the extended real numbers.

1. Let (f_j) be a sequence of $\overline{\mathbb{R}}$-valued measurable functions on $\mathcal{M}$.
   (a) Prove that $g_1(x) = \sup_j f_j(x)$ and $g_2(x) = \inf_j f_j(x)$ are measurable.
   (b) Prove that if $\lim_{j\to\infty} f_j(x)$ exists for every $x \in X$, then f is measurable.

2. Let $f \in L^1(X)$ and $A_n = \{x \in X \,\big|\, |f(x)| \geq n\}$. Prove that $\lim_{n\to\infty} \int_{A_n} |f| = 0$.

3. Suppose $\mu(X) < \infty$. If f and g are $\mathbb{C}$-valued measurable functions on X, you may assume that $\rho(f,g) = \int \frac{|f-g|}{1+|f-g|}\, d\mu$ is a metric. Prove that $f_n \to f$ with respect to this metric if and only if $f_n \to f$ in measure.

4. Let μ be counting measure on $(X, \mathcal{P}(X))$. If $f : X \to [0,\infty]$, prove that
$$\int_X f\, d\mu = \sum_{x\in X} f(x) \left(= \sup_{F \text{ finite}} \sum_{x\in F} f(x) \right).$$
   The last equality is by definition and need not be proved.

5. Define $\lambda : \mathcal{M} \to [0,\infty]$ as $\lambda(A) = \sup\{\mu(B) \,|\, B \subset A, B \in \mathcal{M}, \text{ and } \mu(B) < \infty\}$.
   (a) Show that λ is a measure on $(X, \mathcal{M})$.
   (b) Show that if μ is σ-finite, then $\lambda = \mu$.

6. Consider the following possible definitions of the total variation $|\nu|$ of the complex measure ν:
   (a) $|\nu|(A) = \sup\left\{ \sum_{j=1}^{n} |\nu(A_j)| \,\Big|\, (A_j)_{j=1}^n \text{ is a finite partition of } A \text{ into measurable sets} \right\}$.
   (b) If $\nu(A) = \int_A f\, d\mu$ for some $f \in L^1(\mu)$, then $|\nu|(A) = \int_A |f|\, d\mu$.

   Prove that (a) implies (b).

7. Let D be a Lebesgue measurable subset of $[a,b]$ such that $m(D) > 0$. Prove that for every $\lambda \in [0,1]$, there exists $c \in [a,b]$ such that $m(D \cap [a,c]) = \lambda m(D)$. (Hint: Consider $\int_{[a,b]} \chi_{D\cap[a,x]}\, dm$ for $x \in [a,b]$.)

8. Let $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ be the product measure space of two σ-finite measure spaces. Let f be a nonnegative $\overline{\mathbb{R}}$-valued $\mathcal{M} \otimes \mathcal{N}$-measurable function on $X \times Y$. Prove that for $1 \leq p < \infty$,
$$\left(\int_X \left[\int_Y f(x,y)\, d\nu(y) \right]^p d\mu(x) \right)^{1/p} \leq \int_Y \left[\int_X f(x,y)^p\, d\mu(x) \right]^{1/p} d\nu(y).$$

9. Let X and Y be Banach spaces.

   (a) Let (T_n) be a sequence of bounded linear transformations from X to Y and T be another bounded linear transformation from X to Y. Suppose that for each $x \in X$, $T_n(x) \to T(x)$. Prove that $\sup_n ||T_n|| < \infty$.

   (b) Prove that every weakly convergent sequence in X is bounded in norm.