Real Analysis **Preliminary Examination** May 2015

Do 7 of the following 9 problems. You must clearly indicate which problems are to be graded. If you do not do this, then problems 1-7 will be graded. Strive for clear and detailed solutions. $(X, \mathcal{M}, \mu)$ denotes a measure space. m_n denotes Lebesgue measure on $\mathbb{R}^n$. m denotes Lebesgue measure on $\mathbb{R}$. $\overline{\mathbb{R}}$ denotes the extended real numbers.

1. (a) (6 pts.) Suppose μ is complete. Prove that if $f : X \to \mathbb{R}$ is measurable and $f = g$ μ-a.e., then g is measurable.

   (b) (4 pts.) Suppose μ is not complete. Give an example of a measurable function f and a function g such that $f = g$ μ-a.e., but g is not measurable.

2. Let $1 < p < \infty$, $\frac{1}{p}+\frac{1}{q} = 1$. If $f \in L^p(\mathbb{R}^1)$ and $g \in L^q(\mathbb{R}^1)$, prove that $\lim_{n\to\infty} \int_{\mathbb{R}} f(x)g(nx)\,dx = 0$.

3. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous.

   (a) (6 pts.) Prove that $\lim_{n\to\infty} \int_0^1 f(x^n)\,dx$ exists and evaluate the limit.

   (b) (4 pts.) Does the limit always exist if f is only assumed to be integrable? Justify your answer.

4. Let A be an open subset of $\mathbb{R}^2$. For $h > 0$, define $B_h = \{(x, y, h) \in \mathbb{R}^3 \,|\, (x, y) \in A\}$. Define $C = \{(\lambda x, \lambda y, \lambda z) \,|\, (x, y, z) \in B_h, 0 \le \lambda \le 1\}$. Show that $m_3(C) = \frac{1}{3}hm_2(A)$. You need **not** prove that C is measurable.

5. Suppose $E \subset \mathbb{R}$ is a Lebesgue measurable set such that $m(E) < \infty$. Prove that $\lim_{x\to\infty} m(E \cap (E + x)) = 0$.

6. Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of the signed measure ν.

   (a) (5 pts.) Prove that if λ and μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \ge \nu^+$ and $\mu \ge \nu^-$.

   (b) (5 pts.) Prove that if ν_1, ν_2 are signed measures that both omit the same value $+\infty$ or $-\infty$, then $|\nu_1 + \nu_2| \le |\nu_1| + |\nu_2|$ (where $|\nu|$ denotes total variation of ν).

7. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces and μ be a measure on $\mathcal{M}$. Let $f : X \to Y$ be an $(\mathcal{M}, \mathcal{N})$-measurable mapping. Define $\nu = \mu f^{-1}$ on $\mathcal{N}$ by $\nu(B) = \mu f^{-1}(B) = \mu(f^{-1}(B))\ \forall B \in \mathcal{N}$.

   (a) (2 pts.) Show that ν is a measure.

   (b) (8 pts.) Let $g : Y \to \overline{R}$ be measurable. You may assume $g \circ f$ is measurable. Show that $\int_X g \circ f\,d\mu = \int_Y g d(\mu f^{-1})$ in the sense that the existence of one of the two integrals implies that of the other and the equality of the two.

8. Let $f : [0, 1] \to \mathbb{R}$ be continuous and bounded variation. Prove that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left| f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)\right|^2 = 0.$$

   (Hint: Consider the total variation function of f.)

9. Let X be a normed vector space and (x_n) a sequence in X such that $(f(x_n))$ is Cauchy for all $f \in X^*$. Prove that (x_n) is bounded in norm. (Hint: Consider $J : X \to X^{**}$ such that $(Jx)(y) = y(x)$ for all $y \in X^*$.)