Real Analysis Preliminary Examination

August, 2016

Do 7 of the following 9 problems. You must clearly indicate which 7 are to be graded. If you do not do this, then problems 1-7 will be graded. Strive for clear and detailed solutions.

1. Let μ^* be an outer measure on X induced by a premeasure μ_0 on a semialgebra $\mathcal{A}$ of subsets of X. Prove that for any $E \subset X$, there exists a μ^*-measurable set $A \supset E$ such that $\mu^*(E) = \mu(A)$.

2. Let E be a bounded Lebesgue measurable subset of $\mathbb{R}$ such that the sets $\{E_r = \{x+r : x \in E\} : r \in [0,1] \cap \mathbb{Q}\}$ are disjoint. Prove that $m(E) = 0$.

3. Let X be a uncountable set, and ρ be a set function on $\mathcal{P}(X)$ defined by

$$\rho(E) = \begin{cases} 0, & \text{if } E \text{ has less than 2 elements,} \\ 1, & \text{otherwise.} \end{cases}$$

   Describe the outer measure μ^* on $\mathcal{P}(X)$ generated by ρ, and the σ-algebra $\mathcal{M}$ of all μ^*-measurable sets. Justify your answers.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space, and $f : X \to \mathbb{R}$ be measurable function. Prove that if $\int_E |f| \, d\mu \leq \mu(E)$ for every $E \in \mathcal{M}$, then $-1 \leq f \leq 1$ a.e.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $E = \{(x, y) \in \mathbb{R}^2 : y = f(x)\}$ be the graph of f. Prove that E is Borel measurable on $\mathbb{R}^2$ and $(m \times m)(E) = 0$.

6. Let $\{f_n\}$ be a sequence of measurable functions on $(X, \mathcal{M}, \mu)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. Suppose that there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all n and for all $x \in X$. Prove that for all E with $\mu(E) < \infty$,

$$\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu.$$

7. Let ν be a signed measure on a measurable space $(X, \mathcal{M})$. Prove that

$$\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{M}, F \subset E\}$$

   and

$$\nu^-(E) = -\inf\{\nu(F) : F \in \mathcal{M}, F \subset E\}$$

   for every $E \in \mathcal{M}$.

8. Let f be absolutely continuous on $[a, b]$ and $f' = 0$ a.e. with respect to the Lebesgue measure. Prove that f is a constant on $[a, b]$.

9. Let X be a normed vector space over $\mathbb{R}$ and x be a nonzero vector in X. Prove that there is a closed subspace M of X such that

$$M \cap \operatorname{span}(x) = \{0\} \quad \text{and} \quad X = M + \operatorname{span}(x).$$