Real Analysis **Preliminary Examination** May 2017

Do 7 of the following 9 problems. You must clearly indicate which problems are to be graded. If you do not do this, then problems 1-7 will be graded. Strive for clear and detailed solutions. Notation: $(X, \mathcal{M}, \mu)$ denotes a measure space. 1_A denotes the characteristic function of the set A. $|\nu|$ denotes the total variation of a complex measure.

1. (a) Let $f : D \to [-\infty, \infty]$ be measurable on $D \in \mathcal{M}$. Prove that $\forall \alpha \in [-\infty, \infty]$, $\{x \in D \,|\, f(x) = \alpha\} \in \mathcal{M}$.
   (b) Give an example of a function f defined on $(0,1)$ that is not measurable, but such that $\{x \in (0,1) \,|\, f(x) = \alpha\}$ is Lebesgue measurable $\forall \alpha \in [-\infty, \infty]$.

2. We say that $\{A_\lambda \,|\, \lambda \in \Lambda\} \subset \mathcal{M}$ is almost disjoint if λ_1, $\lambda_2 \in \Lambda$ and $\lambda_1 \neq \lambda_2$ imply that $\mu(A_{\lambda_1} \cap A_{\lambda_2}) = 0$. Prove that if $\{A_n \,|\, n \in \mathbb{N}\}$ is almost disjoint, then
$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

3. Let μ be a nonzero measure on the σ-algebra of all subsets of $\mathbb{R}$ such that μ only takes the values 0 and 1. Prove that μ must be a point mass (i.e. $\exists x_0 \in \mathbb{R}$ such that $\mu(E) = 1_E(x_0)$). (There is more than one way to do this, but a hint for one method is to use some combination of the theorems that describes all complex Borel measures and all positive Borel measures on $\mathbb{R}$. A hint for another method is to use a nested intervals argument.)

4. Let $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ be two increasing functions. Prove that
$$\int_{[a,b]} \beta(x^+) d\mu_\alpha(x) + \int_{[a,b]} \alpha(x^-) d\mu_\beta(x) = \alpha(b^+)\beta(b^+) - \alpha(a^-)\beta(a^-).$$
(Note: The measure induced by an increasing function $G(x)$ is the same as the measure induced by its right continuous modification $G(x^+)$.) (Hint: Use some form of the Tonelli or Fubini Theorem.)

5. Suppose that $f_k \to f$ in $L^p(X)$, $1 \leq p \leq \infty$, $g_k(x) \to g(x) \quad \forall x \in X$, and $\|g_k\|_\infty \leq M \quad \forall k$. Prove that $f_k g_k \to fg$ in $L^p(X)$.

6. Prove that if ν_1 and ν_2 are complex measures on $(X, \mathcal{M})$ such that ν_1 and ν_2 are mutually singular, then $|\nu_1 + \nu_2| = |\nu_1| + |\nu_2|$.

7. Let X be a Banach space and let $J : X \to X^{**}$ such that $J(x) = \hat{x}$, where $\hat{x}(f) = f(x)$, $\hat{x} : X^* \to \mathbb{C}$. Prove that $J(X)$ is a closed subspace of X^{**}. (Just prove that it is closed.)

8. Sketch a proof that $L^p(\mathbb{R}^n)$ has a countable dense subset.

9. Suppose $f : [a,b] \to \mathbb{R}$ is continuous and maps sets of Lebesgue measure zero to sets of Lebesgue measure zero. Prove that f maps Lebesgue measurable sets to Lebesgue measurable sets. (Hint: Continuous functions map compact sets to compact sets.)