Real Analysis **Preliminary Examination** August 2017

Do 7 of the following 9 problems. You must clearly indicate which problems are to be graded. If you do not do this, then problems 1-7 will be graded. Strive for clear and detailed solutions. Notation: $(X, \mathcal{M}, \mu)$ denotes a measure space. $\mathcal{L}^n$ denotes the σ-algebra of Lebesgue measurable sets in $\mathbb{R}^n$, m_n denotes Lebesgue measure on $\mathbb{R}^n$. (if $n = 1$ or if the context is clear, $\mathcal{L}$ and m will be used.) 1_A denotes the characteristic function of the set A. $|\nu|$ denotes the total variation of a complex measure. $\overline{\mathbb{R}}$ denotes the extended real numbers.

1. If $E \in \mathcal{L}$ and $m(E) > 0$, then $\forall \alpha < 1$, prove that there exists an open interval I such that $m(E \cap I) > \alpha m(I)$.

2. Prove that if $\{A_n \mid n \in \mathbb{N}\} \subset \mathcal{M}$ is such that $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ and $\mu(A_n) < \infty\ \forall n \in \mathbb{N}$, then $\{A_n \mid n \in \mathbb{N}\}$ has the property that if $n_1 \neq n_2$, then $\mu(A_{n_1} \cap A_{n_2}) = 0$.

3. Compute the limit and justify your answer:
$$\lim_{n\to\infty} \int_0^{\infty} \frac{n \sin(x/n)}{x(1+x^2)} dx.$$

4. Consider the product measure space $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ of two σ-finite measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$. Let f be an extended $\mathbb{R}$-valued μ-integrable function on X and g an extended $\mathbb{R}$-valued ν-integrable function on Y. Let $h(x, y) = f(x)g(y)$ (with the convention that $0 \cdot (\pm\infty) = 0$).
   (a) Show that h is $\mathcal{M} \otimes \mathcal{N}$-measurable on $X \times Y$.
   (b) Show that h is $(\mu \times \nu)$-integrable on $X \times Y$ and that
$$\int_{X\times Y} h d(\mu \times \nu) = \left(\int_X f d\mu\right)\left(\int_Y g d\nu\right).$$

5. Prove that if $0 \leq j < n$, every j-dimensional hyperplane H in $\mathbb{R}^n$ is a Borel set and $m_n(H) = 0$. (A general hyperplane H can be represented as $A + x$, where A is a vector subspace.)

6. Let X and Y be Banach spaces. If $T : X \to Y$ is a linear map such that $\forall f \in Y^*$ $f \circ T \in X^*$, prove that T is bounded.

7. Let ν be a signed measure on $(X, \mathcal{M})$. You may assume $L^1(\nu) = L^1(|\nu|)$.
   (a) Prove that if $f \in L^1(\nu)$, then $\left|\int f d\nu\right| \leq \int |f| d\nu$.
   (b) Prove that if $E \in \mathcal{M}$, then
$$|\nu|(E) = \sup\left\{\left|\int_E f d\nu\right| \,\Big|\, |f| \leq 1\right\}.$$

8. Assume $f \in L^{p_0}(X)$ for some $0 < p_0 < \infty$. Prove that $\lim_{p \to 0} \int_X |f|^p \, d\mu = \mu\left(\left\{x \middle| f(x) \neq 0\right\}\right)$.
(Hint: Consider $0 < |f(x)| \leq 1$ and $|f(x)| \geq 1$.)

9. Let (f_i), f be $\mathbb{R}$-valued functions on $[a, b]$ such that $\lim_{i \to \infty} f_i(x) = f(x) \quad \forall x \in [a, b]$. Prove that $T_f[a, b] \leq \liminf_{i \to \infty} T_{f_i}[a, b]$. ($T_g[a, b]$ denotes the total variation of g on $[a, b]$.)