Real Analysis Preliminary Examination

May, 2018

Do 7 of the following 9 problems. You must clearly indicate which 7 are to be graded. If you do not do this, then problems 1-7 will be graded. Strive for clear and detailed solutions.

1. Let μ^* be an outer measure on a σ-algebra $\mathcal{M}$ over X. Prove that if $\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$ for any disjoint E and F in $\mathcal{M}$, then μ^* is a measure on $\mathcal{M}$.

2. Let μ^* be an outer measure on $\mathcal{P}(X)$. Prove that for any set $E \subset X$ with $\mu^*(E) < \infty$, if there is a μ^*-measurable set $A \subset E$ such that $\mu^*(A) = \mu^*(E)$, then E itself is μ^*-measurable.

3. Let $(X, \mathcal{M})$ be a measurable space and $\{f_i : i \in \mathbb{N}\}$ be a sequence of $\mathcal{M}$-measurable functions on X. Prove directly that $g(x) = \inf_n f_n(x)$ is $\mathcal{M}$-measurable (**Measurability of** $\sup_n f_n(x)$ **can not be used in your proof.**)

4. Let $A = \{r_n : n = 1, 2, \ldots, \}$ be an enumeration of all rational numbers in $[0, 1]$,
$$F(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{[r_n, \infty)}, \qquad \text{and} \qquad g(x) = \begin{cases} \frac{1}{2^n}, & x = r_n, \\ 0, & \text{otherwise.} \end{cases}$$
Accepting without proof that F is increasing and right continuous, prove that the Lebesgue-Stieltjes measure μ_F generated by F has the properties that $\mu_F(\mathbb{R}) = 1$, $\mu_F(\{r_n\}) = \frac{1}{2^n}$, $\mu_F(A^c) = 0$, and
$$\int g(x)\, d\mu_F = \frac{1}{3}.$$

5. Let f be a real-valued integrable function on a measure space $(X, \mathcal{M}, \mu)$, and $\{E_i\}_{i=1}^{\infty}$ be measurable subsets of X such that
$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = 0.$$
Show that
$$\lim_{n\to\infty} \int_{\bigcap_{i=1}^{n} E_i} f\, d\mu = 0.$$

6. Let $T : \mathbb{R}^n \to \mathbb{R}^k$ be a bounded linear operator. Prove that the graph $\Gamma(T)$ of T is a Borel subset of $\mathbb{R}^{n+k}$ with Lebesgue measure zero.

7. Let ν and μ be **σ-finite positive** measures on a measurable space $(X, \mathcal{M})$ with $\nu \ll \mu$. Show that the Radon-Nikodym derivative $\dfrac{d\nu}{d\mu}$ of ν with respect to μ satisfies
$$\frac{d\nu}{d\mu} \geq 0 \qquad \mu\text{-a.e.}$$

8. Let X be a normed vector space over $\mathbb{R}$ and M be a finite dimensional subspace. Prove that there is a closed subspace N of X such that
$$N \cap M = \{0\} \quad \text{and} \quad X = N + M.$$

9. Let H be a Hilbert space and S be a subset of H. Show that
$$S^{\perp} := \{x \in H : \quad \langle x, s\rangle = 0 \text{ for all } s \in S\}$$
is a closed subspace.