Real Analysis Preliminary Examination

August, 2018

Do 7 of the following 9 problems. You must clearly indicate which 7 are to be graded. If you do not do this, then problems 1-7 will be graded. Strive for clear and detailed solutions.

1. Let $(X,\mathcal{M},\mu)$ be a non-σ-finite measure space. Define
$$\mathcal{N} = \{E \in \mathcal{M} : \text{either } E \text{ is } \sigma\text{-finite or } E^c \text{ is } \sigma\text{-finite for } \mu\}$$
and a set function ν on $\mathcal{N}$ by $\nu(E) = \begin{cases} 0, & \text{if } E \text{ is } \sigma\text{-finite for } \mu, \\ 1, & \text{if } E^c \text{ is } \sigma\text{-finite for } \mu. \end{cases}$ Prove that $\mathcal{N}$ is a σ-algebra and ν is a measure on $\mathcal{N}$.

2. Let μ^* be an outer measure on $\mathcal{P}(X)$, A be a μ^*-measurable set such that $\mu^*(A) < \infty$, and $E \subset A$ be a non-μ^*-measurable set such that $\mu^*(A) = \mu^*(E)$. Prove that $\mu^*(B) = 0$ for any μ^*-measurable $B \subset A \setminus E$.

3. Let $(X,\mathcal{M})$ be a measurable space and $D \subset \mathbb{R}$ be a dense subset of $\mathbb{R}$. Prove that if for a function $f : X \to \mathbb{R}$, $\{x \in X : f(x) < a\} \in \mathcal{M}$ for every $a \in D$, then f is $\mathcal{M}$-measurable.

4. Let F be the extended Cantor function defined by
$$F(x) = \begin{cases} 0, & \text{if } x \le 0, \\ \left(\sum_{i=1}^{n-1} \frac{a_i/2}{2^i}\right) + \frac{1}{2^n}, \text{ where } n = \min\{j : a_j = 1\}, & \text{if } x = \sum_{i=1}^{\infty} \frac{a_i}{3^i},\ a_i \in \{0,1,2\}, \\ 1, & \text{if } x \ge 1. \end{cases}$$
and $C \subset [0,1]$ be the Cantor set. Accepting without proof that F is well defined, increasing, and continuous, prove that $\mu_F(C) = 1$ and $\mu_F \perp m$, where μ_F is the Lebesgue-Stieltjes measure generated by F and m is the Lebesgue measure.

5. Let $(X,\mathcal{M},\mu)$ be a measure space and $f \in L^1(\mu)$. Prove that $E = \{x : |f| > 0\}$ is σ-finite, and there exist $E_1 \subset E_2 \subset \cdots$ with $\mu(E_n) < \infty$ for all n, such that $\lim_{n\to\infty} \int_{E_n} f\, d\mu = \int f\, d\mu$.

6. Let $A = \{a_{ij} \in \mathbb{R}^2 : i,j = 1,2,\ldots\}$ be a subset of $\mathbb{R}^2$ such that $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} |a_{ij}| < \infty$, and $\{S_n : n = 1,2,\ldots\}$ be a partition of $\mathbb{N}^2$ (i.e. $\bigcup_{n=1}^{\infty} S_n = \mathbb{N}^2$ and $S_n \cap S_m = \emptyset$ for all $m \ne n$). Prove that
$$\sum_{n=1}^{\infty} \sum_{(i,j)\in S_n} a_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

7. Let ν_1, ν_2, μ be signed measures with at least one of the ν_1, ν_2 being finite. Prove that if $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$, then $\nu_1 - \nu_2 \perp \mu$.

8. Prove that a linear functional f on a normed vector space X is bounded if and only if $N(f) = \{x : f(x) = 0\}$ is closed.

9. Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, $K(x,t)$ be a Lebesgue measurable function on $\mathbb{R}^2$ such that
$$M := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |K(x,t)|^q\, dt\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} < \infty.$$
Prove that for any $f \in L^p(\mathbb{R})$, $K(x,t)f(t) \in L^1(\mathbb{R}(t))$ for a.e. x, and the operator T defined by
$$T(f)(x) := \int_{\mathbb{R}} K(x,t)f(t)\, dt$$
is linear and bounded operator on $L^p(\mathbb{R})$, and
$$||T|| \le M.$$