Real Analysis Preliminary Exam
May 2019

Directions: Complete exactly seven (7) of the following nine problems, and indicate in the boxes below which seven problems should be graded.

If you do not indicate exactly seven problems in the boxes above, up to seven worked problems will be graded in the order they appear below. Strive for clear, concise, and legible solutions. If a reader cannot easily follow your argument you may not receive credit for that problem. If any problems are nearly identical to an unnamed textbook result, you should assume that you're being asked to prove that result.
Throughout, we follow the usual notational conventions in Folland's textbook: $\mathcal{B}_X$ is the collection of Borel sets on X, $\mathcal{L}$ is the collection of Lebesgue-measurable subsets of $\mathbb{R}$, m^n is the Lebesgue measure on $\mathbb{R}^n$, and $\mathcal{M}$ is an unspecified σ-algebra on an appropriate set. In various contexts, we use the notations $L^p(X, \mathcal{M}, \mu) = L^p(X, \mu) = L^p(\mu)$ interchangeably.

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1. Suppose that $(X, \mathcal{M}, \mu)$ is a semifinite measure space; that is, for every $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{M}$ with $F \subset E$ and $0 < \mu(F) < \infty$. Let $E \in \mathcal{M}$ with $\mu(E) = \infty$. Show that for every $c > 0$ there exists $F \in \mathcal{M}$ with $F \subset E$ and $c < \mu(F) < \infty$.

2. Show that for every $f \in C([0,1])$ and $\epsilon > 0$, there exists an absolutely continuous function g on $[0,1]$ such that $\|f - g\|_u < \epsilon$.

3. (i) Show that if $f \in L^1([0,1], m)$ then $F(x) = \int_{[0,x]} f \, \mathrm{d}m$ is continuous on $[0,1]$.
(ii) Show that the map $I : L^1([0,1], m) \longrightarrow C([0,1])$ given by

$$I(f)(x) = \int_{[0,x]} f \, \mathrm{d}m,$$

is continuous with respect to the usual norms on $L^1([0,1], m)$ and $C([0,1])$.

4. Suppose $A \in \mathcal{L}$ with $m(A) < \infty$ and $\{f_n\} \subset L^1(\mathbb{R}, \mathcal{L}, m)$ converging uniformly on A to f. Show that $f \in L^1(A)$ and $\lim \int_A f_n = \int_A f$.

5. Suppose X, Y and Z are Banach spaces and $T \in L(X, Y)$. Then T induces a map $\widetilde{T} : L(Y, Z) \longrightarrow L(X, Z)$ given by $\widetilde{T}(F) = F \circ T$. Show that $\widetilde{T} \in L(L(Y, Z), L(X, Z))$.

6. Suppose H is an infinite dimensional Hilbert space. Show that the unit sphere $S = \{x \in H \ : \ \|x\| = 1\}$ in H is not compact.
Hint: Consider an orthonormal basis of H and a well-chosen open cover of S.

7. Let $(X, \mathcal{M}, \mu)$ be a measure space.
(i) Show that the subspace $B(X) \subset L^1(\mu)$ of bounded functions is dense in $L^1(\mu)$.
(ii) Give an example which shows that $B(X)$ need not be complete.

8. Suppose $f, g \in L^1(\mathbb{R}, \mathcal{L}, m)$ and $h(x, y) = f(x)g(y - x)$. Show that $h \in L^1(m \times m)$.

9. Let μ be a signed measure on $(X, \mathcal{M})$ and let $\nu = \nu^+ - \nu^-$ be its Jordan decomposition. Show that for all $E \in \mathcal{M}$, $\nu^+(E) = \sup\{\nu(A) \ : \ A \in \mathcal{M}, A \subset E\}$.