Real Analysis Preliminary Exam
August 2019

Directions: Complete exactly seven (7) of the following nine problems, and indicate in the boxes below which seven problems should be graded.

If you do not indicate exactly seven problems in the boxes above, up to seven worked problems will be graded in the order they appear below. Strive for clear, concise, and legible solutions. If a reader cannot easily follow your argument you may not receive credit for that problem. If any problems are nearly identical to an unnamed textbook result, you should assume that you're being asked to prove that result.

Throughout, we follow the usual notational conventions in Folland's textbook: $\mathcal{B}_X$ is the collection of Borel sets on X, $\mathcal{L}$ is the collection of Lebesgue-measurable subsets of $\mathbb{R}$, m^n is the Lebesgue measure on $\mathbb{R}^n$, and $\mathcal{M}$ is an unspecified σ-algebra on an appropriate set. In various contexts, we use the notations $L^p(X, \mathcal{M}, \mu) = L^p(X, \mu) = L^p(\mu)$ interchangeably.

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1. Let $F(x) = \lfloor 2x \rfloor$, where $\lfloor y \rfloor$ is the greatest integer not exceeding y, and let μ_F be the associated Lebesgue-Stieltjes measure. Find $\displaystyle\int_{(0,\infty)} 3^{-x}\,\mathrm{d}\mu_F$.

2. Suppose that μ and ν are σ-finite measures on $(X, \mathcal{M})$ and that $\int f\,\mathrm{d}\mu \leq \int f\,\mathrm{d}\nu$ for all $f \in L^1(\mu) \cap L^1(\nu)$. Show that $\mu = \nu$.

3. Let $F \subset \mathbb{R}$. An element $x \in \mathbb{R}$ is called a *boundary point* of F if every open set $\mathcal{O}$ containing x has $\mathcal{O} \cap F \neq \emptyset$ and $\mathcal{O} - F \neq \emptyset$. Give an example of a closed subset $F \subset [0,1]$ for which the set of boundary points has positive Lebesgue measure.

4. Let $E = [0,1] \times [0,1]$, and $f_n(x,y) = \left(x + \frac{1}{n}\right)^n \left(y - \frac{1}{n}\right)^n$ for each $n \in \mathbb{N}$. Find $\lim_{n\to\infty} \int_E f_n\,\mathrm{d}m^2$.

5. Let μ_1, μ_2 be measures on $(X, \mathcal{M})$.
   (i) Show that $\mu_1 + \mu_2$ is a measure on $(X, \mathcal{M})$.
   (ii) Show that if $f \in L^1(\mu_1 + \mu_2)$ is real-valued, then $\int f\,\mathrm{d}(\mu_1 + \mu_2) = \int f\,\mathrm{d}\mu_1 + \int f\,\mathrm{d}\mu_2$.

6. Let $g \in L^\infty(\mathbb{R}, \mathcal{L}, m)$ and $1 \leq p < \infty$. Define a linear operator T on $L^p(\mathbb{R}, \mathcal{L}, m)$ by $Tf = fg$. Show that T is bounded and $\|T\| = \|g\|_\infty$.

7. (i) Show that Lebesgue measure on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ is translation-invariant.
   (ii) Prove or disprove: every measure on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ is translation-invariant.

8. Let $f \in L^+(\mathbb{R}, \mathcal{B}_\mathbb{R}, m)$ and let c be a nonzero real number. Show that $\int f(cx)\,\mathrm{d}x = \frac{1}{|c|}\int f(x)\,\mathrm{d}x$.

9. Suppose $1 \leq p < \infty$. Show that if $L^p([0,1], m)$ is a Hilbert space, then $p = 2$.
Hint: Apply the Parallelogram Law to $f = \chi_{[0,1/2]}$ and $g = \chi_{(1/2,1]}$.