Real Analysis Preliminary Exam
August 2020

Directions: Complete exactly seven (7) of the following nine problems, and indicate in the boxes below which seven problems should be graded.

If you do not do this, then Problems 1-7 will be graded. Strive for clear, detailed, and legible solutions.

Notation: $(X, \mathcal{M}, \mu)$ denotes a measure space and 1_A denotes the characteristic function of A.

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1. Let $\mu(X) = 1$ and suppose $(A_n) \subset \mathcal{M}$ such that $\mu(A_n) = 1$ for all n. Prove that $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = 1$.

2. Compute $\lim_{n\to\infty} \int_0^{\infty} \frac{x}{1+x^n}\,\mathrm{d}x$, and justify all of your steps.

3.Let (f_n) be a sequence of nonnegative $\mathbb{R}$-valued measurable functions on X such that $f_n \longrightarrow f$ pointwise and $\int f = \lim \int f_n < \infty$. Prove that for all $E \in \mathcal{M}$,

$$\int_E f = \lim \int_E f_n.$$

4. Prove that $L^1(\mathbb{R})$ (Lebesgue measure) has a countable dense subset.

5. If μ and ν are finite positive measures on $(X, \mathcal{M})$, show that there exists a nonnegative measurable function f on X such that for all $E \in \mathcal{M}$,

$$\int_E (1-f)\,\mathrm{d}\mu = \int_E f\,\mathrm{d}\nu.$$

Hint: Consider $\mu + \nu$.

6. Let $(X, \mathcal{M}, \lambda)$ be a signed measure space and let $E \in \mathcal{M}$.

   (a) (8 pts) Show that if $\lambda(E) > 0$ then there exists a positive set E_0 for λ such that $E_0 \subset E$ and $\lambda(E_0) \geq \lambda(E)$

   (b) (2 pts) Show that if $\lambda(E) < 0$ then there exists a negative set E_0 for λ such that $E_0 \subset E$ and $\lambda(E_0) \leq \lambda(E)$.

7. Let $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f_n, f \in L^p$ and $g_n, g \in L^q$. Prove that if $\lim_{n\to\infty} \|f_n - f\|_p = 0$ and $\lim_{n\to\infty} \|g_n - g\|_q = 0$ then $\lim_{n\to\infty} \|f_n g_n - fg\|_1 = 0$.

8. Prove that every weakly convergent sequence in a Banach space X is norm bounded.

9. Let m_2 denote Lebesgue measure on $\mathbb{R}^2$. If $p(x, y)$ is a nonzero polynomial with real coefficients and $Z = \{(x, y) : p(x, y) = 0\}$, prove that $m_2(Z) = 0$.