Real Analysis Preliminary Examination

August, 2021

Direction: Complete seven (7) of the following nine problems, and indicate in the box below which seven problems should be graded. If you do not do this, then problems 1-7 will be graded. Strive for clear and detailed solutions.

1. Let μ^* be an outer measure on X and $E \subset X$. Prove that if for all $\epsilon > 0$, there is a μ^* measurable set $A \subset E$ such that $\mu^*(E \setminus A) < \epsilon$, then E is μ^* measurable.

2. For a fixed x_0, let $F(x) = \chi_{[x_0,\infty)}(x)$ and $(\mathbb{R}, \mathcal{M}, \mu_F)$ be the (complete) Lebesgue-Stieltjes measure space generated by F. Prove that $\mathcal{M} = \mathcal{P}(\mathbb{R})$, i.e., prove that every subset of $\mathbb{R}$ is μ_F^*-measurable.

3. Let μ be σ-finite positive measure and ν be a signed measure on $(X, \mathcal{M})$, respectively. Prove that if $|\nu(E)| \leq \mu(E)$ for all $E \in \mathcal{M}$, then $\nu \ll \mu$, ν is also σ-finite, and
$$\left|\frac{d\nu}{d\mu}\right| \leq 1 \qquad \mu - \text{a.e.}$$

4. Let m be the Lebesgue measure on $\mathbb{R}$. Prove that
$$\lim_{k\to\infty} \lim_{n\to\infty} \int_{E\cap(-k,k)} x^{\frac{2}{n}}\ dx = m(E)$$
for all Lebesgue measurable set E.

5. Let f be a Lebesgue integrable function over $[0, \infty)$ and n be a natural number. Prove that
$$\int_0^\infty \int_{\sqrt[n]{y}}^\infty \frac{f(x)}{x^n}\ dxdy = \int_0^\infty f(x)\ dx.$$

6. Let F be an increasing and absolutely continuous function on $\mathbb{R}$ and μ_F be the Lebesgue-Stieltjes measure generated by F. Prove that
$$\mu_F(E) = \int_E F'\ dm$$
for every Borel set E.

7. Let ν be a signed and μ be a positive measures on a σ-finite measurable space $(X, \mathcal{M})$, respectively, $\nu \ll \mu$, and $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of ν. Prove that
$$\frac{d\nu^+}{d\mu} = \left(\frac{d\nu}{d\mu}\right)^+, \qquad \frac{d\nu^-}{d\mu} = \left(\frac{d\nu}{d\mu}\right)^- \qquad \mu\text{-a.e.}$$

8. Let X be a normed vector space and $f : X \to \mathbb{R}$ be a linear functional. Prove that f is bounded if and only if $f^{-1}(\{0\})$ is closed in X.

9. Let $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$ and $(\mathbb{R}, \mathcal{L}(\mathbb{R}), m)$ be the Lebesgue measure space. For any $f \in L^p(\mathbb{R}, \mathcal{L}(\mathbb{R}), m), g \in L^q(\mathbb{R}, \mathcal{L}(\mathbb{R}), m)$ and any fix $x \in \mathbb{R}$, prove that the function
$$t \mapsto f(x-t)g(t)$$
is a function in $L^1(\mathbb{R}, \mathcal{L}(\mathbb{R}), m)$, and the convolution defined by
$$f * g(x) = \int f(x-t)g(t)\ dt$$
satisfies
$$\|f * g\|_\infty \leq \|f\|_p \|g\|_q.$$