Real Analysis Preliminary Exam
January 2021

Directions: Complete exactly seven (7) of the following nine problems, and indicate in the boxes below which seven problems should be graded.

If you do not do this, then Problems 1-7 will be graded. Strive for clear, detailed, and legible solutions.

Notation: $(X, \mathcal{M}, \mu)$ denotes a measure space.

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1. Prove that if (A_n) is a sequence of measurable sets and $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ then

$$\mu\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n\right) = 0.$$

2. Let X be a topological space and let $\mathcal{F}$ be a family of continuous real-valued functions on X. Prove that the function $g(x) = \sup\{f(x) \;:\; f \in \mathcal{F}\}$ is a Borel measurable function on X.
(Note: $\mathcal{F}$ need not be countable).

3. Let f be an integrable function on X. Prove that for all $\epsilon > 0$ there exists a set E of finite measure such that $\int_{E^c} |f| < \epsilon$.

4. If $0 < q < p + 1$, find, with proof,

$$\lim_{n\to\infty} \int_0^1 \frac{nx^p + x^q}{x^p + nx^q} \,\mathrm{d}x.$$

5. Let (u_j) be a sequence of nonnegative measurable functions. If $u_j \le u$ for all j, where u is a nonnegative measurable function and $\int u \,\mathrm{d}\mu < \infty$, prove that

$$\limsup_{j\to\infty} \int u_j \,\mathrm{d}\mu \le \int \limsup_{j\to\infty} u_j \,\mathrm{d}\mu.$$

6. Let ν be a signed measure on $(X, \mathcal{M})$. Prove that $|\nu|(E) = \sup\{\int_E f \,\mathrm{d}\nu \;:\; |f| \le 1\}$.
($|\nu|$ denotes the total variation of ν.)

7. For $j = 1, 2$, let μ_j, ν_j be σ-finite measures on $(X_j, \mathcal{M}_j)$ such that $\nu_j \ll \mu_j$. Prove that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\frac{\mathrm{d}(\nu_1 \times \nu_2)}{\mathrm{d}(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu_1}(x_1)\frac{\mathrm{d}\nu_2}{\mathrm{d}\mu_2}(x_2).$$

8. Let Z be a vector subspace of the normed space X. Prove that if $y \in X$ has distance $d = \inf_{z \in Z} \|z - y\|$ from Z, then there exists a linear functional $\Lambda : X \longrightarrow \mathbb{R}$ such that $\|\Lambda\| \leq 1$, $\Lambda(y) = d$, and $\Lambda(z) = 0$ for all $z \in Z$.

9. Let $f \in L^\infty(\mu) \cap L^p(\mu)$ for some $0 < p < \infty$. Prove that

$$\|f\|_\infty = \lim_{q \to \infty} \|f\|_q.$$

(Hint: Consider $\limsup$ and $\liminf$.)