**Real Analysis Preliminary Examination**

May, 2021

*Direction:* Complete seven (7) of the following nine problems, and indicate in the box below which seven problems should be graded. If you do not do this, then problems 1-7 will be graded. Strive for clear and detailed solutions.

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1. Let $\mu^*$ be an outer measure on $X$. Prove that for any $E \subset X$, if there exists a $\mu^*$-measurable set $A$ such that $\mu^*(E \triangle A) = 0$, then $E$ is $\mu^*$-measurable.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space such that for each $E \in \mathcal{M}$, either $\mu(E) = 0$ or $\mu(E^c) = 0$. Prove that for any measurable function $f$, either $f \equiv \|f\|_\infty$ or $f \equiv -\|f\|_\infty$ a.e.

3. Prove that for any nonnegative Lebesgue measurable function $f$, $\lim_{n\to\infty} \int_0^\infty \frac{f(x)}{\sqrt[n]{1+x}}\, dx = \int_0^\infty f(x)\, dx$.

4. Let $f$ be an integrable function over a measure space $(X, \mathcal{M}, \mu)$. Prove that if $\left| \int_E f \, d\mu \right| \le \mu(E)$ for all $E \in \mathcal{M}$, then $|f| \le 1$ a.e.

5. Let $m^n$ be the Lebesgue measure over $\mathbb{R}^n$ and $S$ be a set of polynomials over $\mathbb{R}^n$. Define

$$V(S) = \{x \in \mathbb{R}^n : f(x) = 0, \forall f \in S\}$$

(it is called a Zariski closed set in $\mathbb{R}^n$). Prove that if $S$ contains at least one nonzero polynomial, then $m^n(V(S)) = 0$ .

*Hint: If for $x_n = a$, $f(\cdot, a)$ is a zero polynomial over $\mathbb{R}^{n-1}$, then $x_n - a$ is a factor of $f$.*

6. Let

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \left(\sum_{i=1}^{n-1} \frac{a_i/2}{2^i}\right) + \frac{1}{2^n}, & \text{if } x = \sum_{i=1}^{\infty} \frac{a_i}{3^i} \in [0,1], \text{ where } n = \min\{j : a_j = 1\} \text{ with} \\ & \qquad \text{the convention that } \min \emptyset = \infty, \\ 1 & \text{if } 1 < x \end{cases}$$

be the extended Cantor function, $\mu_F$ be the Lebesgue-Stieltjes measure generated by $F$, and $m$ be the Lebesgue measure. Prove that

$$\mu_F \perp m$$

*(Hint: $F(x)$ is continuous and the complement of the Cantor set is a disjoint union of*

$$\left\{ (-\infty, 0), (1, \infty), \left( \sum_{i=1}^{n-1} \frac{a_i}{3^i} + \frac{1}{3^n}, \sum_{i=1}^{n-1} \frac{a_i}{3^i} + \frac{2}{3^n} \right), a_i \in \{0, 2\}, n = 1, 2, \ldots, \right\}.$$

7. Prove that if $f \in BV[a, b]$ for every closed and bounded interval $[a, b]$, then $f$ is Borel measurable.

8. Let $X$ be a normed vector space over $\mathbb{R}$ and $X^*$ be the set of all bounded linear functionals on $X$. Prove that if $f(x) = f(y)$ for all $f \in X^*$, then $x = y$.

9. Let $X, Y$ be normed vector spaces over $\mathbb{R}$ and $L(X, Y)$ be the vector space of all bounded linear operators from $X$ to $Y$. Prove that for each fixed $x \in X$, the map $\hat{x} : L(X, Y) \to Y$ defined by

$$\hat{x}(T) = T(x)$$

is a bounded linear operator, and

$$\|\hat{x}\| \le \|x\|.$$