Real Analysis Preliminary Examination

August, 2022

Direction: Complete seven (7) of the following nine problems, and indicate in the box below which seven problems should be graded. If you do not do this, then problems 1-7 will be graded. Strive for clear and detailed solutions.

1. Let μ^* be an outer measure on X and $E \subset X$. Prove that if for all $\epsilon > 0$, there is a μ^* measurable set $A \subset E$ such that $\mu^*(E \setminus A) < \epsilon$, then E is μ^*-measurable.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and f, g be real valued measurable functions on X such that $g(x) \neq 0$ $\forall x \in X$. Prove that f/g is measurable.

3. Let $\mathcal{A}$ be a semi-algebra over X, μ be a pre-measure on $\mathcal{A}$, and μ^* be the outer measure on X induced by μ.

   a) Prove that for every $E \subset X$, there is a μ^*-measurable $A \supset E$ such that
$$\mu^*(E) = \mu^*(A).$$

   b) Prove that $B \subset X$ is μ^*-measurable if and only if for every μ^*-measurable set A with $\mu^*(A) < \infty$,
$$\mu^*(A) = \mu^*(A \cup B) + \mu^*(A \cup B^c).$$

4. Let $(X, \mathcal{M}, \mu)$ be a measure space, and $f(x)$ be a real valued measurable function on X such that
$$\int_X |f| \, d\mu = 0.$$
   Prove that $f = 0$ a.e.

5. Compute (with justification) $\displaystyle\lim_{n\to\infty} \int_0^\infty \frac{\cos x}{x^{1/n} + x^n} \, dx.$

6. Let $S \subset \mathbb{R}^2$ be the region defined by
$$S = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq \sin(x), 0 < x < \pi\}.$$
   For any $f \in L^1((0, \pi), m)$, prove that $\csc(x) f(x) \in L^1(S, m \times m)$ and
$$\int_S \csc(x) f(x) \, d(m \times m) = \int_0^\pi f(x) \, dx.$$

7. Prove in an infinite dimensional Hilbert space H, the unit ball
$$B = \{h \in H : \|h\| \leq 1\}$$
   is not compact.

8. Let X and Y be Banach spaces over $\mathbb{R}$. Prove that if a bounded linear operator $T : X \to Y$ is one-to-one and onto, then $T^{-1} : Y \to X$ is a bounded LINEAR operator.

9. Let $0 < p, q, r < \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Prove that for any measurable functions f, g over $(X, \mathcal{M}, \mu)$,
$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$