Real Analysis Preliminary Examination

August, 2023

Direction: Complete seven (7) of the following nine problems, and indicate in the box below which seven problems should be graded. If you do not do this, then problems 1-7 will be graded. Strive for clear and detailed solutions.

1. Let $X \neq \emptyset$ and $\mathcal{P}$ be its power set. Define $\mathcal{M} = \mathcal{P}(X)$. Then it is a fact that for any $f : X \mapsto [0, \infty]$,

$$\mu : \mathcal{P}(X) \mapsto [0, \infty] \text{ defined by } \mu(E) = \sum_{x \in E} f(x)$$

is a measure (you do not have to prove this fact). Prove that μ is semifinite if and only if $f(x) < \infty$ for all $x \in X$.

2. Let $X \neq \emptyset$, $\mathcal{P}$ be its power set, and $\mu^* : \mathcal{P}(X) \mapsto [0, \infty]$ be an outer measure on X.

   1. Suppose that $E \subset X$ and $\mu^*(E) = 0$. Prove that E is μ^*-measurable.
   2. Let M denote the collection of μ^*-measurable sets and μ the restriction of μ^* to M. Assuming that μ is a measure, prove that it is complete.

3. Let $X \neq \emptyset$ and $\mathcal{M}$ be a σ-algebra on X. Recall that

$$L^+ = \{f : X \mapsto [0, \infty] \text{ such that } f \text{ is } \mathcal{M}\text{-measurable}\}.$$

Let $f \in L^+$ such that $\int f < \infty$. Prove that $\{x : f(x) = \infty\}$ is a null set and $\{x : f(x) > 0\}$ is σ-finite.

4. Let $X \neq \emptyset$, $\mathcal{M}$ be a σ-algebra on X, and ν be a complex measure on $(X, \mathcal{M})$. Prove that

   1. $L^1(\nu) = L^1(|\nu|)$,
   2. if $f \in L^1(\nu)$, then

$$\left| \int f d\nu \right| \leq \int |f| d|\nu|.$$

5. Let μ be a positive measure. A collection of functions $\{f_\alpha\}_{\alpha \in A} \subset L^1(\mu)$ is said to be uniformly integrable if $\forall \epsilon > 0, \exists \delta > 0$ such that $|\int_E f_\alpha d\mu| < \epsilon$ for all $\alpha \in A$ whenever $\mu(E) < \delta$. Prove that any finite subset of $L^1(\mu)$ is uniformly integrable.

6. Recall that for $F : \mathbb{R} \mapsto \mathbb{C}$, the total variation of F at x is defined by

$$T_F(x) = \sup \left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : n \in \mathbb{N}, -\infty < x_0 < x_1 < \cdots < x_n = x \right\}.$$

Define

$$F(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that $F \in BV([0, 1])$ by proving that $T_F(1) - T_F(0) < 3$.

7. Let $\mathcal{H}$ be a Hilbert space. Fix $y \in \mathcal{H}$ and define $f_y \in \mathcal{H}^*$, the dual space of $\mathcal{H}$, by

$$f_y(x) = \langle x, y \rangle.$$

Prove that

$$\|f_y\|_{\mathcal{H}^*} = \|y\|_{\mathcal{H}}.$$

8. Given a measure space $(X, \mathcal{M}, \mu)$, we denote $L^p(X, \mathcal{M}, \mu)$ by L^p for all $p \in [1, \infty]$. Suppose that $1 \leq p_j \leq \infty$, $n \in \mathbb{N}$ is arbitrary,

$$\sum_{j=1}^{n} p_j^{-1} = r^{-1} \leq 1,$$

and $f_j \in L^{p_j}$ for all $j \in \{1, \ldots, n\}$. Prove that $\prod_{j=1}^{n} f_j \in L^r$ and $\|\prod_{j=1}^{n} f_j\|_{L^r} \leq \prod_{j=1}^{n} \|f_j\|_{L^{p_j}}$.

9. Given a measure space $(X, \mathcal{M}, \mu)$, we denote $L^p(X, \mathcal{M}, \mu)$ by L^p for all $p \in [1, \infty]$. Let $q \in [1, \infty)$. Suppose that $\{f_n\}_{n=1}^{\infty} \subset L^\infty \cap L^1$ satisfies $f_n \to f$ in L^∞ as $n \to \infty$ while $f_n \to g$ in L^q as $n \to \infty$. Prove that $f = g$ almost everywhere.