**Real Analysis Preliminary Examination**

May, 2023

*Direction:* Complete seven (7) of the following nine problems, and indicate in the box below which seven problems should be graded. If you do not do this, then problems 1-7 will be graded. Strive for clear and detailed solutions.

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1. Let $X \neq \emptyset$ and $\mathcal{M}$ be a $\sigma$-algebra on $X$. Prove that every $\sigma$-finite measure on $(X, \mathcal{M})$ is semifinite.

2. Given a set $X \neq \emptyset$ and its power set $\mathcal{P}(X)$, suppose that $\mu^* : \mathcal{P}(X) \mapsto [0, \infty]$ is the outer measure on $X$ and that $\{A_j\}_{j=1}^{\infty}$ is a collection of disjoint $\mu^*$-measurable sets. Prove that

$$\mu^*(E \cap (\cup_{j=1}^{\infty} A_j)) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j) \quad \forall E \subset X.$$

3. Given any measure space $(X, \mathcal{M}, \mu)$, denote $L^1(X, \mathcal{M}, \mu)$ by $L^1$. Prove that if $f_n, g_n, f, g \in L^1$, satisfy all of

$$\begin{aligned} & f_n \to f, \quad g_n \to g \ \ \mu\text{-a.e.} \\ & |f_n| \leq g_n, \\ & \int g_n d\mu \to \int g d\mu, \end{aligned}$$

then

$$\int f_n d\mu \to \int f d\mu.$$

4. Suppose $f : [0, 1] \mapsto \mathbb{C}$. Prove that

$$|f(x) - f(y)| \leq |x - y| \ \ \forall x, y \in [0, 1] \tag{1}$$

if and only if $f$ is absolutely continuous, differentiable almost everywhere, and $|f'(x)| \leq 1$ almost everywhere.

5. Let $X = [0, 1]$, $\mathcal{M} = \mathcal{B}_{[0,1]}$, $m =$ Lebesgue measure and $\mu =$ counting measure on $\mathcal{M}$. Prove that $m \ll \mu$, but $dm \neq f d\mu$ for any $f \in L^1(\mu)$.

6. The following is a fact that holds true. "Let $\nu$ be a finite signed measure and $\mu$ a positive measure on $(X, \mathcal{M})$. Then $\nu \ll \mu$ implies that for every $\epsilon > 0$, there exists $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $E \in \mathcal{M}$ and $\mu(E) < \delta$." Prove that the claim can fails when $\nu$ is not necessarily a finite measure by giving a counterexample.

7. Suppose that $\mathcal{X}$ is a Banach space, $\mathcal{X}^*$ is its dual space,

$$\begin{aligned} & \Lambda_n \in \mathcal{X}^* \quad \forall\, n \in \mathbb{N}, \\ & \lim_{n \to \infty} \Lambda_n x \text{ exists for all } x \in \mathcal{X}, \\ & \text{and we define } \Lambda x = \lim_{n \to \infty} \Lambda_n x \ \ \forall \ x \in \mathcal{X}. \end{aligned}$$

Prove that $\Lambda \in \mathcal{X}^*$.

8. Consider $X = [0, 2\pi]$ and $L^2([0, 2\pi])$ with Lebesgue measure $m$ and its inner product defined by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f\bar{g} dm(x).$$

Let $e_n(t) = e^{int}$ for $n \in \mathbb{N}$; you may take it for granted that $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for $L^2([0, 2\pi])$. Prove that

$$\lim_{n\to\infty} \int_E e_n(t)dt = 0$$

for all $E \subset [0, 2\pi]$ that is measurable.

9. Recall the definition of

$$L^\infty = L^\infty(X, \mathcal{M}, \mu) = \{f : X \mapsto \mathbb{C} : f \text{ is } \mathcal{M}\text{-measurable and } \|f\|_{L^\infty} < \infty\}$$

where

$$\|f\|_{L^\infty} = \inf\{a \geq 0 : \mu(\{x : |f(x)| > a\}) = 0\}.$$

Prove that the simple functions are dense in $L^\infty$. You may freely use the following two facts. First, $\|\cdot\|_{L^\infty}$ is norm. Second, $\|f_n - f\|_{L^\infty} \to 0$ as $n \to \infty$ if and only if there exists $E \in \mathcal{M}$ such that $\mu(X \setminus E) = 0$ and $f_N \to f$ uniformly on $E$.