Real Analysis Preliminary Examination

August, 2024

***Directions:* Complete seven (7) of the following nine problems, and indicate in the box below which seven problems should be graded.**

If you do not do this, then problems 1-7 will be graded. Strive for clear, detailed, and legible solutions. Notations: $(X, \mathcal{M}, \mu)$ denotes a measure space, m denotes the Lebesgue measure on $\mathbb{R}$, and m_n denotes the Lebesgue measure on $\mathbb{R}^n$.

1. Let $f : X \to \mathbb{R}$ be measurable and $A \in \mathcal{M}$ be nonempty. Prove

   a. $\mathcal{M}_A = \{E \cap A : E \in \mathcal{M}\}$ is a σ-algebra on A, and

   b. $f_A := f \Big|_A$ is $\mathcal{M}_A$-measurable.

2. Let $f : X \to [-\infty, \infty]$ be a measurable function. Let

$$h(x) = \begin{cases} \frac{1}{f(x)}, & \text{if } f(x) \neq 0, \pm\infty, \\ 0, & \text{if } f(x) = \pm\infty, \\ 2024, & \text{if } f(x) = 0. \end{cases}$$

   Prove that h is measurable.

3. Prove that $\lim_{n\to\infty} \int_0^n \left(\frac{\sin x}{x}\right)^n dx = 0$. (*Hints:* $|\sin(x)| < |x|$ *if* $x > 0$, *and* $\int_1^\infty \frac{1}{x}\, dx < \infty$)

4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\Gamma = \{(x, f(x)) : x \in \mathbb{R}\}$. Prove that Γ is measurable and $(m \times m)(\Gamma) = 0$.

5. Let X and Y be Banach spaces and let $\{T_n\}$ be a sequence in $L(X, Y)$ (bounded linear operators) such that $\lim_{n\to\infty} T_n(x)$ exists $\forall\, x \in X$. Let $T(x) = \lim_{n\to\infty} T_n(x)$. Prove that $T \in L(X, Y)$ (*Just show that T is bounded*).

6. If $1 \leq p_1 < p < p_2 \leq \infty$, prove that $L^p(X) \subset L^{p_1}(X) + L^{p_2}(X)$.

7. Let $(Y, \mathcal{N})$ be a measurable space, $f : Y \mapsto [-\infty, \infty]$ be $\mathcal{N}$-measurable, $\phi : X \to Y$ be $(\mathcal{M}, \mathcal{N})$-measurable, $\nu = \mu \circ \phi^{-1}$. You may assume that ν is a measure and $f \circ \phi$ is $\mathcal{M}$-measurable. Prove that $\int_X (f \circ \phi)(x)\, d\mu(x) = \int_Y f(y)\, d\nu(y)$.

8. Suppose μ is a positive measure and $g : X \to [-\infty, \infty]$ is integrable. Let ν be the signed measure defined by $d\nu = g d\mu$. Prove that $d|\nu| = |g| d\mu$ and $|\nu|(A) = \sup\left\{\left|\int_A f\, d\nu\right| : |f| \leq 1\right\}$ $\forall\, A \in \mathcal{M}$ (*recall that* $|\nu| = \nu^+ + \nu^-$).

9. Let $f : (a, b) \to \mathbb{R}$ such that $f'(x)$ exists $\forall\, x \in (a, b)$, and $\exists M \geq 0$ such that $|f'(x)| \leq M$ $\forall\, x \in (a, b)$. Prove that $\forall\, E \subset (a, b)$, $m^*(f(E)) \leq M m^*(E)$.